Графы Ноймайера и их конструкции

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Main source of the talk

The talk is mainly based on the paper:

A general construction of strictly Neumaier graphs and a related switching, `https://arxiv.org/abs/2109.13884`

by

◇ Rhys J. Evans

◇ Sergey Goryainov

◇ Elena V. Konstantinova

◇ Alexander D. Mednykh

Historical background

In: Arnold Neumaier, Regular cliques in graphs and special 1 1/2 designs, In: Finite Geometries and Designs: Proceedings of the Second Isle of Thorns Conference 1980, Eds. P.J. Cameron, J.W.P. Hirschfeld, D.R. Hughes, 1981, 244–259. https://doi.org/10.1017/CBO9781107325579.027

Arnold Neumaier studied:
- regular cliques in edge-regular graphs, and
- a certain class of designs whose point graphs are strongly regular and contain regular cliques.

Question

Does there exist an edge-regular, non-strongly regular graph which contains regular clique?

Historical background

Almost 40 years later, Gary Greaves and Jack Koolen finally proved that such graphs exist, and give two general constructions of graphs:

Gary R. W. Greaves, and Jack H. Koolen, Edge-regular graphs with regular cliques, European Journal of Combinatorics, 342(10) (2019) 2818-2820. https://doi.org/10.1016/j.ejc.2018.04.004

Gary R. W. Greaves, and Jack H. Koolen, Another construction of edge-regular graphs with regular cliques, Discrete Mathematics, 342(10) (2018) 2818-2820. https://doi.org/10.1016/j.disc.2018.09.032.

Historical background

At the end of 2018, it was suggested by Sergey Goryainov to use the following definitions for the graphs under considerations.

Neumaier graph

A non-complete edge-regular graph containing a regular clique.

Strictly Neumaier graph

A non-strongly regular Neumaier graph.

These definitions first appeared in the paper:

Rhys J. Evans, Sergey Goryainov, and Dmitry Panasenko, The smallest strictly Neumaier graph and its generalisation, Electronic Journal of Combinatorics, 26(2) (2019) P2.29. https://doi.org/10.37236/8189

Historical background

After the first constructions of strictly Neumaier graphs were published, there has been an increased interest the study of these graphs.

All currently known constructions (as of 21/09/2022) can be found in the webpage by Rhys J. Evans:

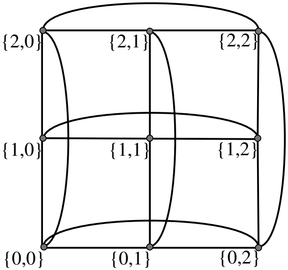
https://rhysje00.github.io/projects/neumaier_graphs

Strongly regular graph vs edge-regular graph

Definition

G is a strongly regular graph if:

- every two adjacent vertices have λ common neighbours (edge-regular graph)
- every two non-adjacent vertices have μ common neighbours (co-edge-regular).

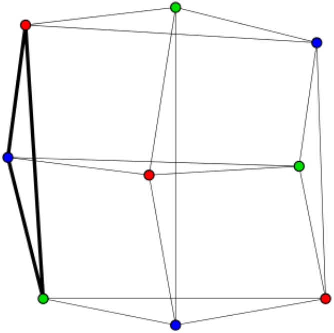


$SRG(9,4,1,2)$: the lattice graph $L_{3,3}$ / 3×3 rook's graph

Regularity of subsets

Definition

A vertex subset $X \subset V$ of a graph $G = (V, E)$ is e-regular if for every $v \notin X$: $|N(v) \cap X| = e$, where e is called the nexus.



3×3 rook's graph is the graph of triangular duoprism

1-regular subset $\cong$ 1-regular clique

Neumaier's question

Theorem (Neumaier, 1981)

A vertex- and edge-transitive graph with a regular clique is strongly regular.

Problem (Neumaier, 1981)

Is a regular, edge-regular graph with a regular clique necessarily SRG?

(strictly) Neumaier graph (2018)

A Neumaier graph is a regular, edge-regular graph with a regular clique. It is a strictly Neumaier graph if it is not strongly regular. A Neumaier graph has parameters $(n, k, \lambda; e, s)$ if it is an edge-regular graph with parameters (n, k, λ), admitting an e-regular clique of size s.

Open questions (2018)

Do strictly Neumaier graphs exist?
For which parameter sets do strictly Neumaier graphs exist?

Known results

Greaves-Koolen (2018) Edge-regular graphs with regular cliques

There are (infinitely many) strictly Neumaier graphs.

Parametrised Cayley graphs.

Evans-Goryainov-Panasenko (2019)

1. The smallest strictly Neumaier graph.
2. There is an infinite class of strictly Neumaier graphs.

Based on affine polar graphs.

Abaid-De Boeck-Koolen-(2020-2021+)

1. An infinite class of Neumaier graphs and non-existence results.
2. Neumaier graphs with few eigenvalues.

Evans-Goryainov machine

Let $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ be edge-regular graphs with parameters (n, k, λ) that admit a partition into perfect 1-codes of size a, where a is a proper divisor of $\lambda + 2$, and $t = \frac{\lambda+2}{a}$. For any $j \in \{1, \ldots, t\}$, let $H_1^{(j)}, \ldots, H_{\frac{n}{a}}^{(j)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(j)}$. Let $\Pi = (\pi_2, \ldots, \pi_t)$ be a $(t-1)$-tuple of permutations from $Sym(\{1, \ldots, \frac{n}{a}\})$.

Denote by $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ the graph obtained as follows.

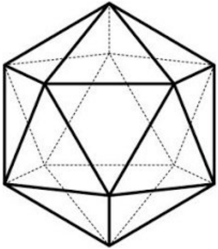
1. Take the disjoint union of the graphs $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$.
2. For any $i \in \{1, \ldots, \frac{n}{a}\}$, connect any two vertices from $H_i^{(1)} \cup H_{\pi_2(i)}^{(2)} \cup \ldots \cup H_{\pi_t(i)}^{(t)}$ to form a 1-regular clique of size at.

Main result (EGKM-2022+)

The graph $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ is a Neumaier graph with parameters $(nt, k + at - 1, \lambda; 1, at)$ whose vertex set admits a partition into 1-regular cliques of size at. Moreover, if $t \geqslant 2$, then $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ is a strictly Neumaier graph.

Examples

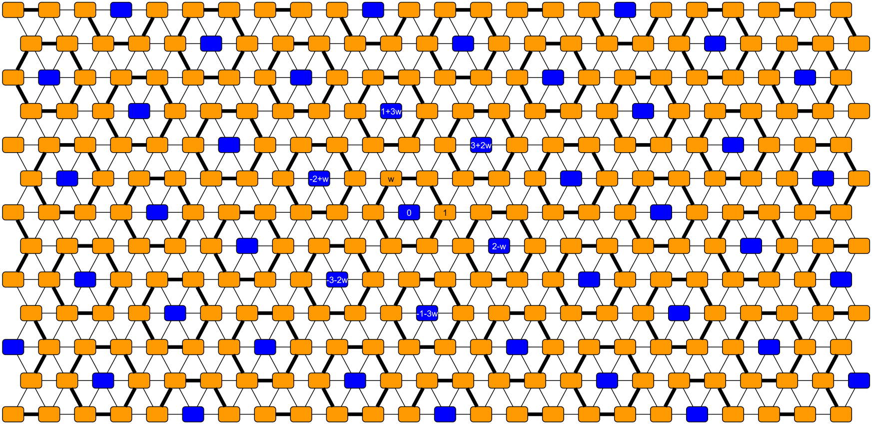
Construction given by a pair of icosahedrons:



The icosahedral graph is an edge-regular graph with parameters $(12, 5, 2)$ that admits a partition into 6 perfect 1-codes of size $a = 2$. Thus, we can use $t = \frac{\lambda+2}{a} = 2$ copies of the icosahedral graph in the general construction to produce four pairwise non-isomorphic strictly Neumaier graphs (depending on the choice of the permutation π_2) with parameters $(24, 8, 2; 1, 4)$.

Construction given by the 6-regular triangular grid

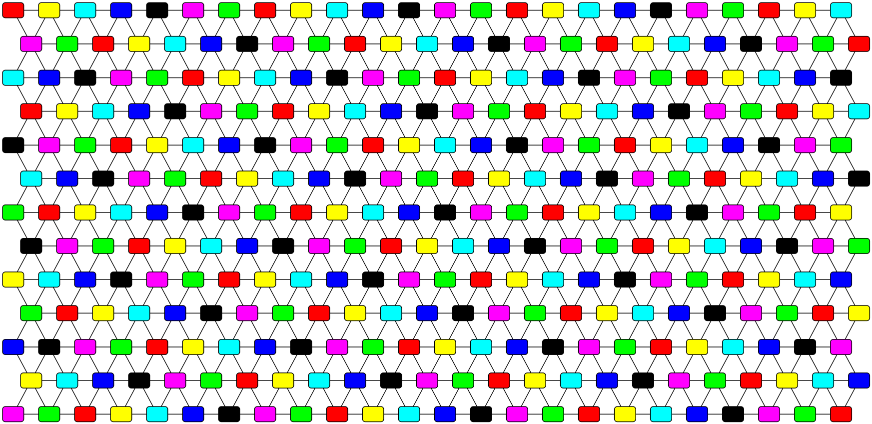
Step 1: find a perfect 1-code



Hint: The ideal I generated by an element of norm 7

Construction given by the 6-regular triangular grid

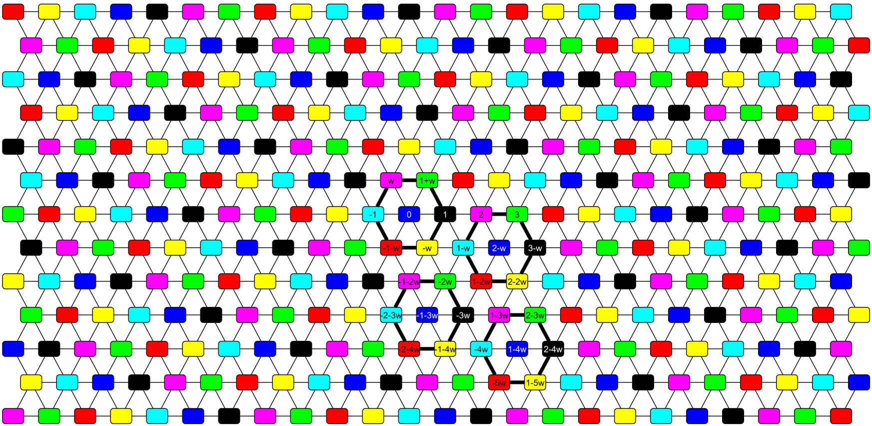
Step 2: find a partition of the triangular grid into 7 perfect 1-codes



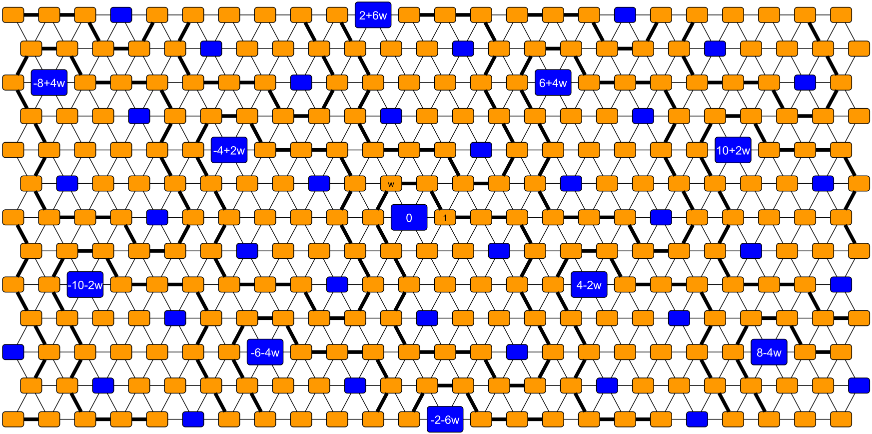
Hint: I is an additive subgroup of index 7 in $\mathbb{Z}[\omega]$

Construction given by the 6-regular triangular grid

Step 3: fix a block of 4 balls of radius 1



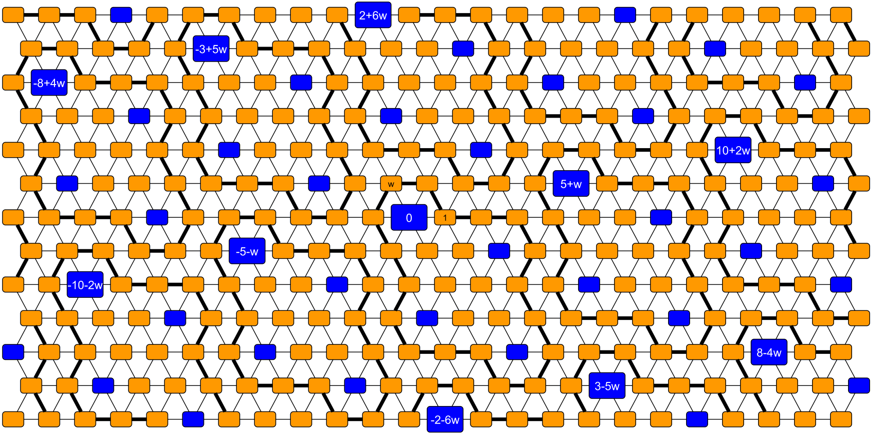
Step 4 – 1: consider a tessellation given an additive subgroup



Hint: additive shifts by $T_1 := \{2(-2+\omega)x + 14y \mid x, y \in \mathbb{Z}\}$

Construction given by the 6-regular triangular grid

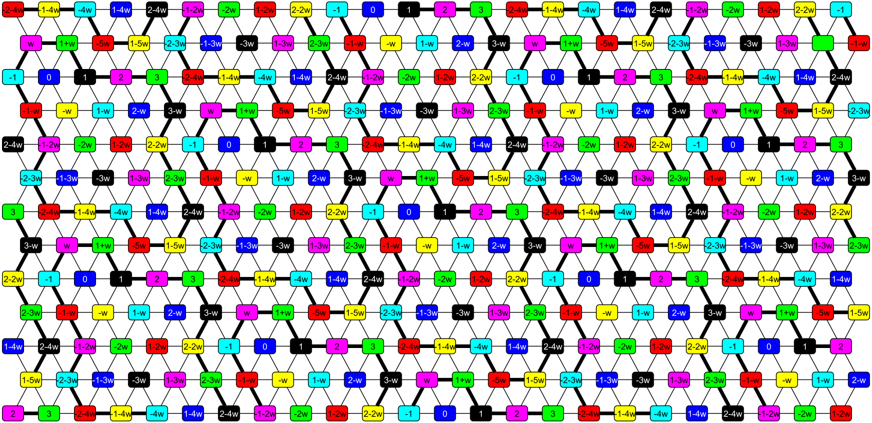
Step 4 – 2: consider a tessellation given an additive subgroup



Hint: additive shifts by $T_2 := \{(5+\omega)x + 28y \mid x, y \in \mathbb{Z}\}$

Construction given by the 6-regular triangular grid

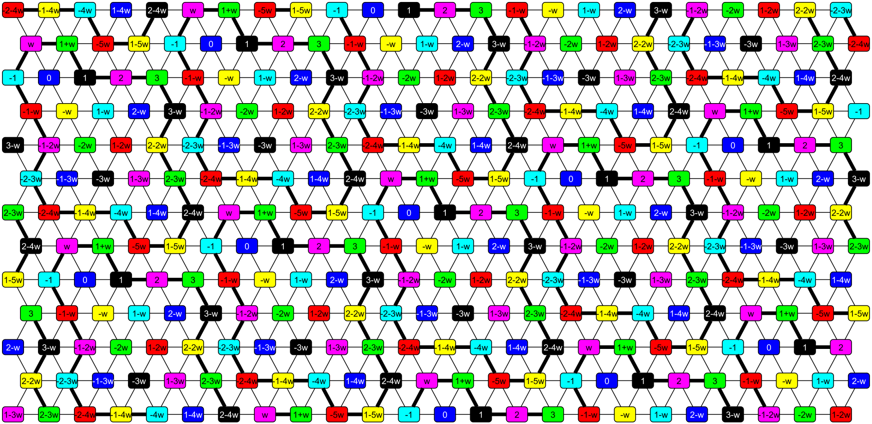
Step 5 – 1: consider a quotient graph Δ_1 of the triangular grid by T_1



Hint: $\Delta_1 := Cay(G_1, \{\pm(1+T_1), \pm(\omega+T_1), \pm(\omega^2+T_1)\})$, where
$G_1 := \mathbb{Z}[\omega]/T_1$

Construction given by the 6-regular triangular grid

Step 5 – 2: consider a quotient graph Δ_2 of the triangular grid by T_2



Hint: $\Delta_2 := Cay(G_2, \{\pm(1+T_2), \pm(\omega+T_2), \pm(\omega^2+T_2)\})$, where $G_2 := \mathbb{Z}[\omega]/T_2$

Construction given by the 6-regular triangular grid

Finally,

• each of the graphs Δ_1 and Δ_2 is edge-regular with parameters $(28, 6, 2)$

• and admits a partition into perfect 1-codes of size $a = 4$;

• these partitions are given by the original partition of the triangular grid into perfect 1-codes;

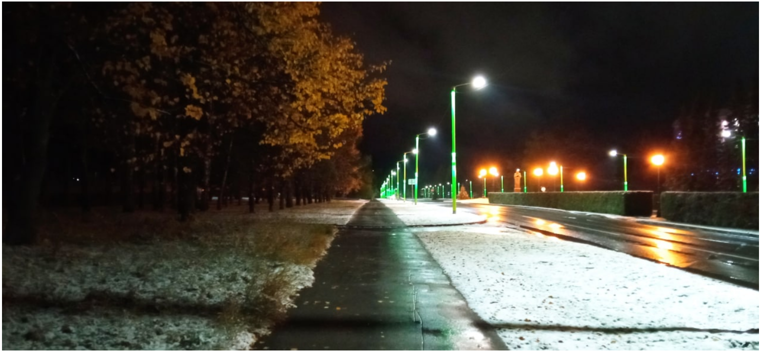
• apply Evans-Goryainov machine and get two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$.

- *n*-dimensional case of the triangular grid?
- other grids? (operation on grids?)
- how one can use root systems?
- generalisation to hyperbolic spaces?

Main problems:

◇ find a perfect code

◇ find a subgroup



Thanks for your attention!