Algebraic and combinatorial approaches to investigating integral graphs

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Outline of talks

The main goal

To present algebraic and combinatorial methods in integral graph theory.

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◇ Short historical background

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- ◇ Classes of graphs:
  - cubic integral graphs

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◇ Classes of graphs:

- cubic integral graphs (a classification is completed)
- quartic integral graphs

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- classical spectral methods

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◇ Combinatorial methods:

- representation theory of finite groups

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- ◇ Applications

Historical background: 1974

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A graph Γ is *integral* if its spectrum consists entirely of integers, where the spectrum of Γ is the spectrum of its adjacency matrix.

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Remark. We believe our bound is far from being tight and the number of integral graphs is substantially smaller.

Computational results on graphs: 1999-2004

K. Balińska, D. Cvetković, M. Lepović, S. Simić, D. Stevanović, M. Kupczyk, K.T. Zwierzyński, G. Royle

- Brendan McKay's program *GENG* for generating graphs
- Magma

Connected intergal graphs with $n \leqslant 12$ vertices

n	2	3	4	5	6	7	8	9	10	11	12
total	2	2^3	2^6	2^{10}	2^{15}	2097152	2^{28}	2^{36}	2^{45}	2^{50}	2^{66}
#	1	1	2	3	6	7	22	24	83	236	325

Simplest examples

Spectrum of the complete graph K_n

$[(-1)^{n-1}, (n-1)^1]$ for $n \geqslant 2$, and $[0^1]$ for $n = 1$. *Integral for any* $n \geqslant 1$.

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Spectrum of n-cycle C_n

The spectrum consists of the numbers $2\cos(\frac{2\pi i}{n})$, $i = 1, \ldots, n$ with multiplicities $2, 1, 1, \ldots, 1, 2$ for n even and $1, 1, \ldots, 1, 2$ for n odd.

There are only three integral cycles:

C_3: $[-1^2, 2]$ $\quad (C_3 \cong K_3)$

C_4: $[-2, 0^2, 2] = [0^2, \pm 2]$ $\quad (C_4 \cong K_{2,2})$

C_6: $[-2, -1^2, 1^2, 2] = [\pm 1^2, \pm 2]$

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C_6: $[-2, -1^2, 1^2, 2] = [\pm 1^2, \pm 2]$

Smallest non-integral cycle is C_5: $[2, (\frac{-1+\sqrt{5}}{5})^2, (\frac{-1-\sqrt{5}}{5})^2]$

Classification of integral cubic graphs: 1975-1978

D. Cvetkovič (1975)

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F. C. Bussemaker, D. Cvetkovič (1976);A.J.Schwenk (1978)

There are exactly 13 connected, cubic, integral graphs.

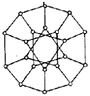
UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 544 — № 576 (1976), 43—48.

552. **THERE ARE EXACTLY 13 CONNECTED, CUBIC, INTEGRAL GRAPHS***

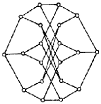
F. C. Bussemaker and *D. M. Cvetković***

1. Results. A graph is called integral if its spectrum consists entirely of integers. Cubic graphs are regular graphs of degree 3.
It was proved in [3] that the set I_r of all connected regular integral graphs of a fixed degree r is finite. At the same time the search for cubic integral graphs was begun. Now we complete this work by the following theorem.

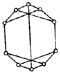
Theorem 1. *There are exactly* 13 *connected, cubic, integral graphs. They are displayed in Fig.* 1 *and in Fig.* 2 *of* [3].

G_9


$3,\ 2^4,\ 1^5,\ -1^5,\ -2^4,\ -3$

G_{10}


$3,\ 2^4,\ 1^5,\ -1^5,\ -2^4,\ -3$

G_{11}


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Classification of integral quartic graphs: 1998-??

Known results

• 1888 possible spectra of 4-regular bipartite integral graphs
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• exhaustive lists of:
- 32 connected 4-regular integral Cayley graphs;
- 27 connected 4-regular integral arc-transitive graphs;
(M. Minchenko, I. M. Wanless, 2015)

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17 quartic bipartite Cayley graphs

n	8	10	12	16	18	24	30	32	36	40	48	72	120
#	1	1	2	1	1	3	1	1	1	1	1	2	1

users.monash.edu.au/~iwanless/data/graphs/IntegralGraphs

How to get not vertex-transitive integral graphs?

Known facts

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Question

Are there any other graph operations preserving the integrality?

Answer

YES: dual Siedel switching!

Which Cayley graphs are integral?: 2009

Cayley graph

Let G be a group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$. In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the action of the generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$.

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Properties

By the definition,
(i) Γ is undirected with no loops;
(ii) Γ is a connected regular graph of degree $|S|$;
(iii) Γ is a vertex–transitive graph.

Classification of integral Cayley graphs

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Characterization of integral Cayley graphs

- Cayley graphs over abelian groups (W. Klotz, T. Sander, 2010) (determine all abelian Cayley integral groups)
- Integral Cayley graphs and groups (A. Ahmady, J. P. Bell, B. Mohar, 2014) (determine all Cayley integral groups)
- Cayley graphs over dihedral groups (L. Lu, Q. Huang, X. Huang, 2018) (determine all integral Cayley graphs over D_p for a prime p)

Definition

A group G is a Cayley integral group if for every symmetric subset S of G, $\Gamma = Cay(G, S)$ is an integral graph.

Integral Cayley graphs over Sym_n

The Star graph $S_n = Cay(\mathrm{Sym}_n, T), \;\; n \geqslant 2$

is the Cayley graph over the symmetric group Sym_n with the generating set $T = \{(1\;i), \; 2 \leqslant i \leqslant n\}$.

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Properties of the Star graph

- connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ (S. B. Akers, B. Krishnamurthy (1989))
- vertex-transitive and edge-transitive
- contains hamiltonian cycles (V. Kompel'makher, V. Liskovets, 1975, P. Slater 1978)
- it does contain even ℓ–cycles where $\ell = 6, 8, \ldots, n!$
- has hierarchical structure
- has integral spectrum

Integral Cayley graphs: the Star graphs S_n

Conjecture (A. Abdollahi and E. Vatandoost, 2009)

The spectrum of S_n is integral, and contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an eigenvalue of S_n).

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The spectrum of S_n contains only integers.

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YES: dual Siedel switching!

W. Haemers (1984): Dual Seidel switching

Dual Seidel switching

For any simple graph Γ with adjacency matrix $A(\Gamma)$ and an order 2 automorphism φ of Γ interchanging only non-adjacent vertices, we have

$$PA(\Gamma)P^T = A(\Gamma),$$

where P is the permutation matrix corresponding to the automorphism φ, and $PA(\Gamma)$ is a symmetric $(0,1)$-matrix with zero diagonal and thus can be viewed as an adjacency matrix of some simple graph.

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Important remark

Note further that $(PA(\Gamma))^2 = (A(\Gamma))^2$. In particular, if Γ is integral, then a graph obtained from Γ by the dual Seidel switching is integral as well.

Dual Siedel switching $\rightarrow$ the Star graphs

Important remark

If Γ is integral $\Rightarrow$ a graph obtained from Γ by the dual Siedel swithing induced by an order 2 automorphism of Γ is integral as well.

Dual Siedel switching $\to$ the Star graphs

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Our goal is to find an appropriate an order 2 automorphism of the Star graph S_n. The automorphism group of S_n is $Aut(S_n) \cong \mathrm{Sym}_n \mathrm{Sym}_{n-1}$, i.e. $|Aut(S_n)| = n!(n-1)!$.

Dual Siedel switching → the Star graphs

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Definitions

Let n be a positive integer, $n \geqslant 3$. Consider the symmetric group $G = \mathrm{Sym}_n$ and put $S = \{(1\ i) \mid i \in \{2, \ldots, n\}\}$. The *left Star graph* (resp. *right Star graph*) is the Cayley graph $Cay_L(\mathrm{Sym}_n, S)$ with left (right) multiplication (resp. $Cay_R(\mathrm{Sym}_n, S)$).

Properties of the left (right) Star graphs

Lemma

Let $L_G = \{\varphi_\pi^\ell \mid \pi \in G\}, R_G = \{\varphi_\pi^r \mid \pi \in G\}$ *be the groups of left and right shifts* G*, respectively. The following statements hold:*
(1) L_G *is a group of automorphisms of* $Cay_L(G, S)$*;*
(2) R_G *is a group of automorphisms of* $Cay_R(G, S)$*;*

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(2) *R_G is a group of automorphisms of $Cay_R(G, S)$;*

Theorem

If $\pi_\ell, \pi_r \in \mathrm{Sym}_n$ satisfies

(1) *π_ℓ, π_r are of order 2;*
(2) *π_ℓ, π_r have different parity;*
(3) *$\pi_r S \pi_r^{-1} = S$;*
(4) *π_ℓ is not conjugate to any element in $\pi_r S$,*

then $\varphi_{\pi_\ell,\pi_r} : x \to \pi_\ell x \pi_r$ is an order 2 automorphism of $Cay_L(\mathrm{Sym}_n, S)$ interchanging only non-adjacent vertices from different parts in bipartition of the left Star graph $Cay_L(\mathrm{Sym}_n, S)$, where $x \in \mathrm{Sym}_n$.

Integral Star graphs and dual Seidel switching

Corollary

For a positive integer $n \geqslant 5$, $\varphi_{(2\ 4),(2\ 3)(4\ 5)}$ *is an order 2 automorphism of the left Star graph* $Cay_L(\mathrm{Sym}_n, S)$ *interchanging only non-adjacent vertices.*

Corollary

For a positive integer $n \geqslant 5$, $\varphi_{(2\ 4),(2\ 3)(4\ 5)}$ is an order 2 automorphism of the left Star graph $Cay_L(\mathrm{Sym}_n, S)$ interchanging only non-adjacent vertices.

A new 4-regular graph

From the Star graph S_5 with the spectrum:

$$[0^{30}, (\pm 1)^4, (\pm 2)^{28}, (\pm 3)^{12}, (\pm 4)^1]$$

by Corollary we have a new 4-regular not vertex-transitive graph with spectrum

$$[-3^7, -2^{13}, -1^3, 0^{15}, 1^1, 2^{15}, 3^5, 4^1]$$

Integral Odd graphs and dual Seidel switching

Definition

For a positive integer m, the *Odd graph*, denoted by O_{m+1}, is the graph whose vertex set is the set of m-subsets of a $(2m+1)$-set X, where two m-sets are adjacent if and only if they are disjoint.

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The Odd graphs are among a more general family of *Johnson graphs*.

Integral Odd graphs and dual Seidel switching

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For a positive integer m, the *Odd graph*, denoted by O_{m+1}, is the graph whose vertex set is the set of m-subsets of a $(2m+1)$-set X, where two m-sets are adjacent if and only if they are disjoint.

The Odd graphs are among a more general family of *Johnson graphs*.

The Odd graphs are not Cayley graphs.

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A new 4-regular graph

In the case $m = 3$, we have two new not vertex-transitive 4-regular not vertex-transitive graphs with spectra

$$\{(-3)^5, (-2)^4, (-1)^9, 1^5, 2^{10}, 3^1, 4^1\}$$

and

$$\{(-3)^4, (-2)^6, (-1)^8, 1^6, 2^8, 3^2, 4^1\}.$$

Integral graphs and dual Seidel switching

S. Goryainov, E. V. Konstantinova, H. Li, D. Zhao, 2020

Integral graphs obtained by dual Seidel switching, Linear Algebra and its Applications, 604 (2020) 476-489.

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Result

In this paper we apply the dual Seidel switching to the Star graphs and to the Odd graphs, which gives two infinite families of integral graphs. In particular, we obtain three new 4-regular integral connected graphs.

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Question

Are there other switchings preserving the integrality?

Dual Seidel switching and Deza graphs

Erickson-Fernando-Haemers-Hardy-Hemmeter-1999

The dual Seidel switching was used for constructing (strictly) Deza graphs from strongly regular graphs.

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Deza graphs, 1994

A Deza graph G with parameters (n, k, b, a) is a k-regular connected graph of order n for which the number of common neighbours of two distinct vertices takes just two values, b or a, where $b \geqslant a$.

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Deza graphs in terms of matrices

Suppose G is a graph with n vertices, and M is its adjacency matrix. Then G is a Deza graph with parameters (n, k, b, a) if and only if

$$M^2 = a\ A + b\ B + k\ I$$

for some symmetric $(0, 1)$-matrices A, B such that $A + B + I = J$.

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Theorem

Let G be a strongly regular graph with parameters (n, k, λ, μ), where $k \neq \mu$, $\lambda \neq \mu$. Let M be the adjacency matrix of G, and P be a non-identity permutation matrix of the same size. Then PM is the adjacency matrix of a Deza graph H if and only if P represents a Seidel automorphism. Moreover, H is a strictly Deza graph if and only if $\lambda \neq 0$, $\mu \neq 0$.

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Question

Are there other ways to get Deza graphs with s.r.g. G_A and G_B (graphs defined by the matrices A and B)?

Generalized dual Seidel switching I

Theorem (Kabanov-K-Shalaginov-2020+)

Let G be a s.r.g. with the adjacency matrix M, and H be its induced subgraph with the adjacency matrix M_{11}. If there exists a Seidel automorphism of H with the permutation matrix P_{11} such that $P_{11}M_{12}M_{22} = M_{12}M_{22}$, then matrices

$$N_1 = \begin{pmatrix} P_{11}M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \text{ and } N_2 = \begin{pmatrix} P_{11}M_{11} & P_{11}M_{12} \\ M_{21}P_{11} & M_{22} \end{pmatrix}$$

are the adjacency matrices of Deza graphs with strongly regular children. Moreover, $N_2^2 = M^2$ and $N_1^2 = (PMP)^2$.

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Simple example

Let $G \cong T(7)$. If take $H = L_2(3)$ with the diagonal symmetry, then there is a Deza graph with parameters $(21, 10, 5, 4)$ whose spectrum is $[10^1, 3^4, 2^3, -2^{11}, -3^2]$.

Generalised dual Seidel switching II

Theorem (Kabanov-K-Shalaginov-2020+)

Let G be a Deza graph with strongly regular children and the adjacency matrix M, and H be its induced subgraph with the adjacency matrix M_{11}. If there exists a Seidel automorphism of H with the permutation matrix P_{11} such that $P_{11}M_{11}M_{12} = M_{11}M_{12}$, then matrices

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More complicated example

Let $G = L_2(m)$ and $H = K_2 \times K_m$, then there is a Seidel automorphism of H corresponding to the central symmetry of the lattice with two rows and m columns, and for $m \geqslant 6$ there are sought integral Deza graphs.

Singular strongly Deza graph

A Deza graph with strongly regular children is called a *strongly Deza graph*.

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Any singular strongly Deza graph is an integral graph with four distinct eigenvalues.

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Examples

- Deza graph with parameters $(12, 6, 3, 2)$ and spectrum $[6^1, 2^3, 0^2, (-2)^6]$
- Deza graph with parameters $(8, 4, 2, 0)$ and spectrum $[4^1, 2^1, 0^3, (-2)^3]$

Theorem *(G. Chapuy and V. Feray,* 2012*)*

The spectrum of the Star graph contains only integers.

Integral Cayley graphs over Sym_n

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Among all sets of $n-1$ transpositions which generate the symmetric group, the set whose associated Cayley graph has the highest λ_2 (the second smallest non-negative eigenvalue) is the set $T = \{(1\ i),\ 2 \leqslant i \leqslant n\}$.

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Corollary

There are no other integral Cayley graphs over the symmetric group generated by sets of $n-1$ transpositions.

Multiplicities of eigenvalues of the Star graphs

Theorem *(G. Chapuy and V. Feray, 2012)*

The multiplicity $mul(n-k)$, $1 \leqslant k \leqslant n-1$, of $(n-k) \in \mathbb{Z}$ is given by:

$$mul(n-k) = \sum_{\lambda \vdash n} dim(V_\lambda) I_\lambda(n-k),$$

where $dim(V_\lambda)$ is the dimension of an irreducible module, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ, satisfying $c(n) = n-k$.

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Theorem *(S. Avgustinovich, E. Khomyakova, E. K., 2016)*

$$\mathrm{mul}(n-2) = (n-1)(n-2)$$
$$\mathrm{mul}(n-3) = \frac{(n-3)(n-1)}{2}(n^2 - 4n + 2)$$
$$\mathrm{mul}(n-4) = \frac{(n-2)(n-1)}{6}(n^4 - 12n^3 + 47n^2 - 62n + 12)$$
$$\mathrm{mul}(n-5) = \frac{(n-2)(n-1)}{24}(n^6 - 21n^5 + 169n^4 - 647n^3 + 1174n^2 - 820n + 60)$$

Multiplicities of eigenvalues of the Star graphs

Theorem *(E. Khomyakova, 2018)*

Let $n, k \in \mathbb{Z}$, $n \geqslant 2$ and $1 \leqslant k \leqslant \frac{n+1}{2}$, then the multiplicity $mul(n-k)$ of the eigenvalue $(n-k)$ of the Star graph S_n is given by the following formula:

$$\mathrm{mul}(n-k) = \frac{n^{2(k-1)}}{(k-1)!} + P(n),$$

where $P(n)$ is a polynomial of degree $2k-3$.

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Catalogue of the Star graph eigenvalue multiplicities (E. Khomyakova, E. Konstantinova, 2019)

Multiplicities $\mathrm{mul}(n-k)$ of eigenvalues $(n-k)$ of the Star graphs S_n for $n \leqslant 50$ and $1 \leqslant k \leqslant n$ are presented in the catalogue. Negative eigenvalues $-(n-k)$ have the same multiplicities as the corresponding positive ones.

n	mul(0)
4	4
5	30
6	168
7	840
8	3960
9	19782
10	150640
11	2089296
12	36011160
13	615154540
14	10058919024
15	158755300080
16	2446623357360
17	37180388161350
18	562723553743200
19	8609968637492640
20	136834037294232600

n	mul(0)
21	2362305285068081220
22	4668364711918838040 0
23	108231799193976661560 0
24	2866940210237670799848 0
25	8235846311096528101791 00
26	245788298238466686153372 48
27	7437339518963013450833112 00
28	2256873385721520138845697880 0
29	684105464925952548262639920792
30	2070129971674121167077493154544 0
31	62595819488086889418818159986518 4
32	189494659230589952145367102001035 20
33	5759808477345846694071637854280986 30
34	17653913968491423747128277755728026 816
35	549111783334822055069672069343534784 320
36	1749199911110957040296760364190367726 5688
37	57760413645503379010832485628805987730 0180

n	mul(0)
38	20045214161520719656501343733468647442343920
39	739952909795026470270714737199811323856785072
40	29222192669334526110964999773556310398591228240
41	1230755917765824096949390167464313250363248267060
42	54702435049128670258626361893397282522722821124000
43	2531180638482250397635439910040738080021778965170400
44	120404540036518230989551268934056697886796380722098640
45	5830994520024240512182674246166203184664150596748260200
46	28558799946024524594550675890724601396141033813989177 1680
47	14087557866064153242858310529196022374457008834526880685600
48	698233161880802136904523173665083589953534868653745159722400
49	34731341207704459607094131312251492828402668161403430943758620
50	1733139483848699201861708583736015380726259651081186948733294400

Combinatorial methods for getting multiplicities

The representation theory of the symmetric group Sym_n

Conjugacy classes of Sym_n are labeled by partitions of n, and the set of inequivalent irreducible representations is defined by partitions of n.

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B. Sagan, 2001

$$\sum_{\lambda\vdash n} (\dim(V_\lambda))^2 = |\mathrm{Sym}_n|$$

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On the Bisection Width of the Transposition Network

Let $n \geqslant 2$ be an integer. Then T_n is an integral Cayley graph:
- the largest eigenvalue of T_n is $n(n-1)/2$ with multiplicity 1;
- the second largest eigenvalue of T_n is $n(n-3)/2$ with multiplicity $(n-1)^2$;
- $n(n-2k+1)/2$ is an eigenvalue of T_n with multiplicity at least $n!/(n(n-k)!(k-1)!)$ for $1 \leqslant k \leqslant n$.

Application: The bisection width problem

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1) $nn!/4$, if n is even.

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Useful reference: X. Liu, S. Zhou, (2019)

Eigenvalues of Cayley graphs, 115 p. (https://arxiv.org/abs/1809.09829)

Eigenfunctions of the Star graphs

S. Goryainov, V. Kabanov, E.K., L. Shalaginov, A. Valyuzhenich, (2020, 2021)

A family of eigenfunctions with non-zero eigenvalues of S_n is obtained. For any eigenvalue $n-m-1$, where $n > 2m$, a connection of these functions with the standard basis of a Specht module for Sym_n is established. For the largest non-principal eigenvalue $n-2$, it is proved that any eigenfunction of the Star graphs S_n can be reconstructed by its values on the second neighbourhood of a vertex.

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V. Kabanov, E.K., L. Shalaginov, A. Valyuzhenich, (2020)

An arbitrary $(n-2)$-eigenfunction of the Star graphs S_n with the minimum cardinality of the support is the difference of the characteristic functions of two completely regular codes of covering radius 2.

THANK YOU FOR ATTENTION!