# Chromatic properties of Cayley graphs

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# Chromatic properties: the chromatic number $\chi$

A mapping $c : V(\Gamma) \to \{1, 2, \ldots, k\}$ is called a *proper $k$–coloring* of a graph $\Gamma = (V, E)$ if $c(u) \neq c(v)$ whenever $u$ and $v$ are adjacent.

*The chromatic number* $\chi = \chi(\Gamma)$ of a graph $\Gamma$ is the least number of colors needed to color vertices of $\Gamma$.

A subset of vertices assigned to the same color forms an independent set, i.e. a $k$–coloring is the same as a partition of the vertex set into $k$ independent sets.

## Known bounds

| | | |
|---|---|---|
| *R. L. Brooks (1941):* | $\chi \leqslant \Delta$ | *(except $K_n$; $C_n$, $n$ is odd)* |
| *P. A. Catlin (1978):* | $\chi \leqslant \frac{2}{3}(\Delta + 3)$ | *(for $C_4$–free graphs)* |
| *A. Johansson (1996):* | $\chi \leqslant O\left(\frac{\Delta}{\log \Delta}\right)$ | *(for $C_3$–free graphs)* |

# Chromatic properties: the chromatic index $\chi'$

*The chromatic index* $\chi' = \chi'(\Gamma)$ of a graph $\Gamma$ is the least number of colors needed to color edges of $\Gamma$ s.t. no two adjacent edges share the same color.

## Known bounds

*V. G. Vizing (1968):* $\Delta \leqslant \chi' \leqslant \Delta + 1$

$\Delta = \Delta(\Gamma)$ is the maximum degree of $\Gamma$

# Chromatic properties: the total chromatic number $\chi''$

In the *total coloring* of a graph $\Gamma$ it is assumed that no adjacent vertices, no adjacent edges, no edge and its endvertices are assigned the same color.

*The total chromatic number* $\chi'' = \chi''(\Gamma)$ of a graph $\Gamma$ is the least number of colors needed in any total coloring of $\Gamma$.

**Known bounds**

*V. G. Vizing (1968):* $\Delta + 1 \leqslant \chi''$ *(from the definition)*

**Total coloring conjecture**

*V. G. Vizing, V. Behzad (1964-1968):* $\chi'' \leqslant \Delta + 2$

# Cayley graphs

Let $G$ be a finite group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$.

In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the action of the generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$.

## Properties

*(i)* $\Gamma$ *is a connected* $|S|$*–regular graph;*
*(ii)* $\Gamma$ *is a vertex–transitive graph.*

## Trivial bounds

| | | |
|---|---|---|
| *From the Brooks' bound:* | $\chi \leqslant \|S\|$ | *(vertex coloring)* |
| *From the Vizing' bound:* | $\chi' = \|S\|$ | *(edge coloring)* |
| *From the Vizing' bound:* | $\|S\| + 1 \leqslant \chi''$ | *(total coloring)* |

# Random Cayley graphs

Let $G$ be a finite group of order $n$, and let $S \subset G$ be a random subset of $G$ obtained by choosing randomly, uniformly and independently (with repetitions) $k \leqslant n/2$ elements of $G$, and by letting $S$ be the set of these elements and their inverses, without the identity. Thus, $|S| \leqslant 2k$.

In the *random Cayley graph* $\Gamma(G, k)$ vertices correspond to the elements of the group and edges correspond to the action of the random $k$ generators.

**Trivial bounds**

*From the Brooks' bound:* $\chi \leqslant 2k + 1$ *(for any finite group G)*

# N. Alon (2013): General results for random Cayley graphs

## General groups

*For any group $G$ of order $n$, and any $k \leqslant n/2$, the chromatic number $\chi(G, k)$ satisfies a.a.s.:*

$$\Omega\left(\left(\frac{k}{\log k}\right)^{1/2}\right) \leqslant \chi(G, k) \leqslant O\left(\frac{k}{\log k}\right)$$

a.a.s.=asymptotically almost surely, i.e.,
the probability it holds tends to 1 as $n$ tends to infinity

We write:
$f = O(g)$, if $f \leqslant c_1 g + c2$ for two functions $f$ and $g$.
$f = \Omega(g)$, if $g = O(f)$.

# N. Alon (2013): General results for random Cayley graphs

## General cyclic groups

*For any fixed $\epsilon > 0$, if $n$ is integer and $1 \leqslant k \leqslant (1-\epsilon)\log_3 n$, the chromatic number $\chi(\mathbb{Z}_n, k)$ for any cyclic group $\mathbb{Z}_n$ satisfies a.a.s.:*

$$\chi(\mathbb{Z}_n, k) \leqslant 3$$

a.a.s.=asymptotically almost surely

# N. Alon (2013): General results for random Cayley graphs

## General abelian groups

*For any abelian group $G$ of size $n$ and any $k \leqslant \frac{1}{4} \log \log(n)$, the chromatic number $\chi(G, k)$ satisfies a.a.s.:*

$$\chi(G, k) \leqslant 3$$

a.a.s.=asymptotically almost surely

# N. Alon (2013): Particular results

## Elementary abelian 2-groups

*For any elementary abelian 2-group* $\mathbb{Z}_2^t$ *of order* $n = 2^t$*, and for all* $k < 0.99 \log_2 n$*, the chromatic number* $\chi(\mathbb{Z}_2^t, k)$ *satisfies a.a.s.:*

$$\chi(\mathbb{Z}_2^t, k) = 2$$

So, for these groups it is typically 2.

# N. Alon (2013): Open questions

- non-abelian case;
- in particular, the symmetric group:

> *"The general problem of determining or estimating more accurately the chromatic number of a random Cayley graph in a given group with a prescribed number of randomly chosen generators deserves more attention. It may be interesting, in particular, to study the case of the symmetric group $Sym_n$."*

N. Alon, The chromatic number of random Cayley graphs, *European Journal of Combinatorics*, 34 (2013) 1232–1243.

# Cayley graphs on the symmetric group $Sym_n$

**L. Babai (1978)**

*Every group has a Cayley graph of chromatic number $\leqslant \omega$; for solvable groups the minimum chromatic number is at most 3.*

$\omega$ is the clique number of a graph (the size of a largest clique).

**R. L. Graham, M. Grötshel, L. Lovász(Eds.) (1995) "Handbook of Combinatorics", Vol.1**

*Every finite group has a Cayley graph of chromatic number $\leqslant 4$.*

Remark: This is a consequence of the fact that every finite simple group is generated by at most 2 elements.

# Bichromatic Cayley graphs on $Sym_n$

## Necessary and sufficient conditions

*Let $\Gamma = Cay(Sym_n, S)$ is a Cayley graph on the symmetric group $Sym_n$. Then $\Gamma$ is bichromatic $\iff$ $S$ does not contain even permutations.*

It follows from the Kelarev's result, which describes all finite inverse semigroups with bipartite Cayley graphs.

A.V. Kelarev, On Cayley graphs of inverse semigroups, *Semigroup forum* 72 (2006) 411–418.

# Bichromatic random Cayley graphs on $Sym_n$

## EK, Kristina Rogalskaya (2015)

*Let a generating set $S$ of a random Cayley graph $\Gamma = Cay(Sym_n, S)$ consists of $k$ randomly chosen generators of $Sym_n$. If $n \geqslant 2$ and $k < \frac{n!}{2}$, then $\Gamma = Cay(Sym_n, S)$ is not, asymptotically almost surely, bichromatic.*

However, these results don't give the conditions for a random Cayley graph $\Gamma$ to be connected.

## Open question

*What are the necessary and sufficient conditions for $\Gamma = Cay(Sym_n, S)$ to be connected, where $S$ is a randomly chosen generating set?*

# Connected Cayley graphs on $Sym_n$

## Question

*What are the necessary and sufficient conditions for $\Gamma = Cay(Sym_n, S)$ to be connected?*

## $T.\ Chen, S.\ Skiena$ (1996)

*Let $S$ of a Cayley graph $\Gamma = Cay(Sym_n, S)$ consists of all reversals of fixed length $\ell$:* $[\pi_1 \ldots \underline{\pi_i \ldots \pi_{i+\ell-1}} \ldots \pi_n]r_l = [\pi_1 \ldots \underline{\pi_{i+\ell-1} \ldots \pi_i} \ldots \pi_n]$.

*Then $\Gamma = Cay(Sym_n, S)$ is connected $\iff \ell \equiv 2 \pmod 4$.*

*In this case $|S| = n - \ell$ and the number of such sets is equal to $\lfloor \frac{n+1}{4} \rfloor$.*

T. Chen, S. Skiena, Sorting with fixed-length reversals, *Discrete applied mathematics,* 71 (1996) 269–295.

# Known connected Cayley graphs on $Sym_n$

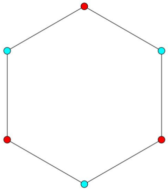
## The Bubble-Sort graph $B_n$

*The Bubble-Sort graph is the Cayley graph on the symmetric group $Sym_n$, $n \geqslant 3$ with the generating set $\{(i\ i+1) \in Sym_n, 1 \leqslant i \leqslant n-1\}$.*

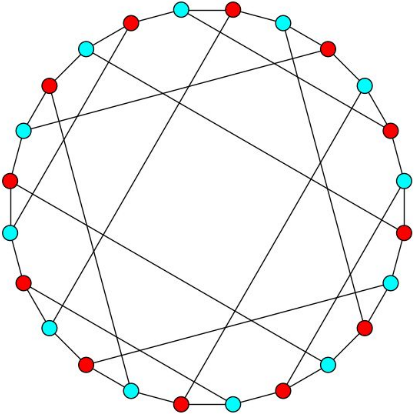
## The Star graph $S_n$

*The Star graph is the Cayley graph on the symmetric group $Sym_n$, $n \geqslant 3$ with the generating set $\{(1\ i) \in Sym_n, 2 \leqslant i \leqslant n\}$.*

Example: $S_3 = Cay(Sym_3, \{(1\ 2), (1\ 3)\} \cong C_6$



# Bichromatic Star graph
# $S_4 = Cay(Sym_4, \{(1\ 2), (1\ 3), (1\ 4)\})$



Picture: Tomo Pisanski

# Connected Cayley graphs on $Sym_n$: the Pancake graph

## The Pancake graph $P_n$

*The Pancake graph is the Cayley graph on the symmetric group $Sym_n$ with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where $r_i$ is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation $\pi$ when multiplied on the right, i.e.,*
$[\underline{\pi_1 \ldots \pi_i}\pi_{i+1} \ldots \pi_n]r_i = [\underline{\pi_i \ldots \pi_1}\pi_{i+1} \ldots \pi_n]$.

## Properties

- *connected*
- $(n-1)$*–regular*
- *vertex–transitive*
- *has a hierarchical structure*
- *is hamiltonian*

# Chromatic properties of the Pancake graph (EK, 2015)

**Total chromatic number**

$\chi''(P_n) = n$ *for any* $n \geqslant 3$.

**Total chromatic index**

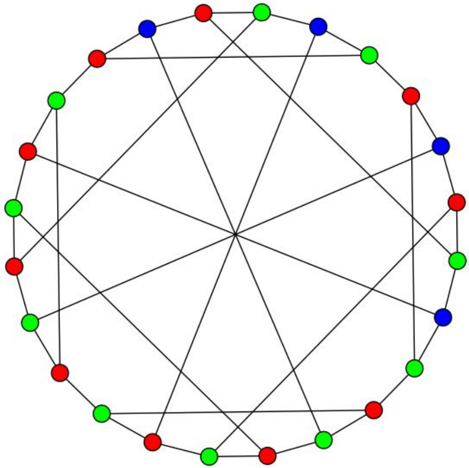
$\chi'(P_n) = n - 1$ *for any* $n \geqslant 3$.

The chromatic index of the Pancake graphs is obtained from Vizing's bound $\chi' \geqslant \Delta$ taking into account the edge coloring, in which the color $(i - 1)$ is assigned to the prefix–reversal $r_i$, $2 \leqslant i \leqslant n$.

**Chromatic number**

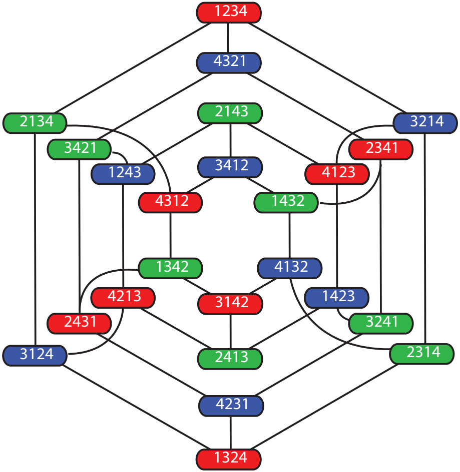
$\chi(P_n) \leqslant n - 2$ *for any* $n \geqslant 5$.

# 3-coloring of $P_4$: hamiltonian drawing



Picture: Tomo Pisanski

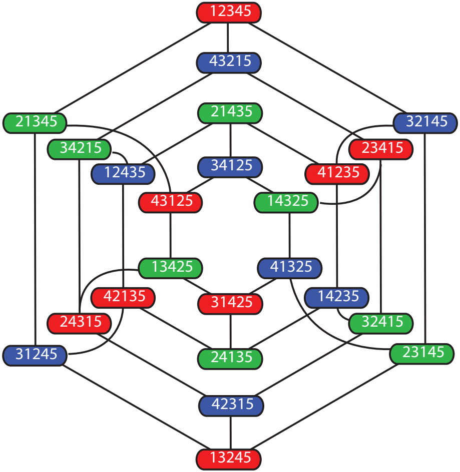
# 3-coloring of $P_4$: hierarchical drawing




Picture: K. Rogalskaya

Idea: A. Williams (2013)

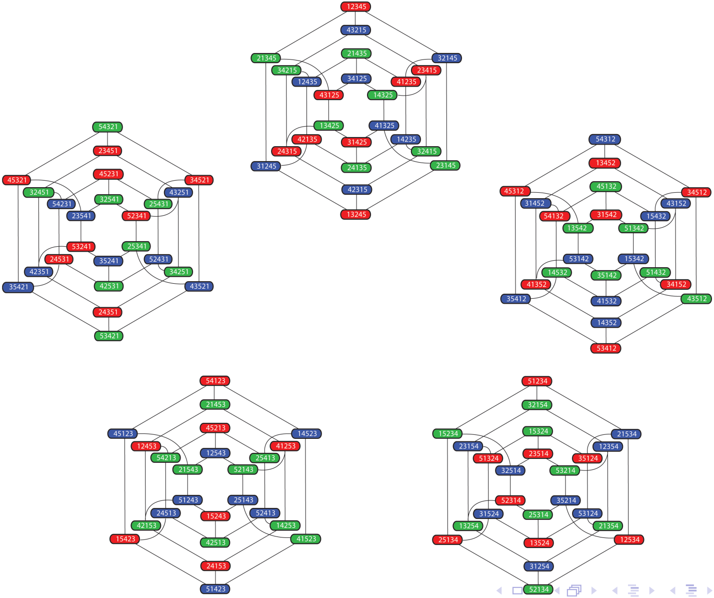
# 3-coloring of one copy of $P_5$: hierarchical drawing




Picture: K. Rogalskaya

Idea: A. Williams (2013)

# 3-coloring $P_5$: hierarchical drawing



# The chromatic number of the Pancake graph (EK, 2015)

## Theorem

*The following holds for $P_n$:*

*1) if $5 \leqslant n \leqslant 8$, then*

$$\chi(P_n) \leqslant \begin{cases} n-k, & \text{if } n \equiv k \pmod 4 \text{ for } k = 1,3; \\ n-2, & \text{if } n \text{ is even;} \end{cases} \tag{1}$$

*2) if $9 \leqslant n \leqslant 16$, then*

$$\chi(P_n) \leqslant \begin{cases} n-(k+2), & \text{if } n \equiv k \pmod 4 \text{ for } k = 1,3; \\ n-4, & \text{if } n \text{ is even;} \end{cases} \tag{2}$$

*3) if $n \geqslant 17$, then*

$$\chi(P_n) \leqslant \begin{cases} n-(k+4), & \text{if } n \equiv k \pmod 4 \text{ for } k = 1,2,3; \\ n-8, & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{3}$$

# Exact values of the chromatic number for $P_n$

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi$ | 2 | 3 | 3 | 4 | 4 | 6? | 6? | 6? | 6? | 6? | 6? | 6? | 6? | 6? | 12? |

$\underline{n = 4, 5}$: examples

$\underline{n = 6}$: Jernej Azarija computed optimal 4-coloring

$\underline{n = 7}$: since $P_{n-1}$ is an induced subgraph of $P_n$,
$\chi(P_7)$ is at least 4, and due to (1) in Theorem we have that $\chi(P_7) = 4$

$\underline{n = 8}$: from (1) in Theorem we have $4 \leqslant \chi(P_8) \leqslant 6$

$\underline{9 \leqslant n \leqslant 16}$: from (2) in Theorem we have $4 \leqslant \chi(P_8) \leqslant 6$