Two graphs: problems and results

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Graphs and problems

Graphs

- *Star graph*
- *Pancake graph*

Problems

- *Hamiltonicicty*
- *Automorphism group*
- *Perfect codes*
- *Diameter*
- *Colouring*

Cayley graphs

Let G be a group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$.

Definition

In the Cayley graph $\Gamma = Cay(G, S) = (V, E)$ *vertices correspond to the elements of the group, i.e.* $V = G$, *and edges correspond to the action of the generators, i.e.* $E = \{(g, gs) : g \in G, s \in S\}$.

The definition of Cayley graph was introduced by A. Cayley in 1878 to explain the concept of abstract groups which are generated by a set of generators in Cayley's time.

Properties

(i) Γ *is a connected regular graph of degree* $|S|$;
(ii) Γ *is a vertex–transitive graph.*

Star and Pancake graphs: definitions

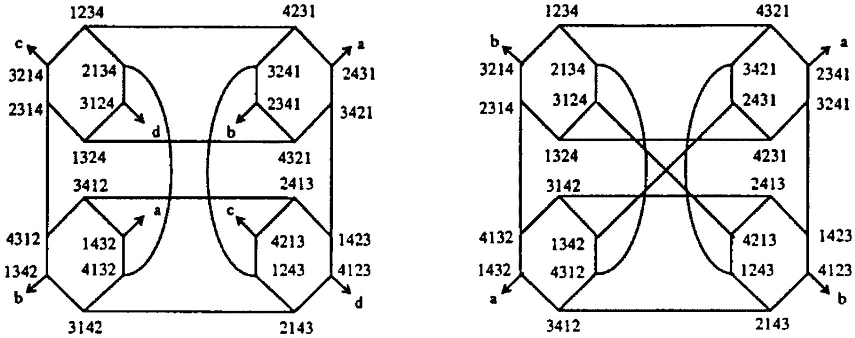
The Star graph S_n

is the Cayley graph on the symmetric group Sym_n with generating set $\{t_i \in Sym_n, 1 \leqslant i < n\}$, where t_i is the operation of transposing the 1st and ith elements, $2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\pi_1\pi_2\dots\pi_{i-1}\pi_i\pi_{i+1}\dots\pi_n]t_i = [\pi_i\pi_2\dots\pi_{i-1}\pi_1\pi_{i+1}\dots\pi_n]$.

The Pancake graph P_n

is the Cayley graph on the symmetric group Sym_n with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\pi_1\dots\pi_i\pi_{i+1}\dots\pi_n]r_i = [\pi_i\dots\pi_1\pi_{i+1}\dots\pi_n]$.

Star and Pancake graphs: examples



(a) (b)

Star and Pancake graphs: properties

Let $\Gamma_n \in \{S_n, P_n\}$.

Properties

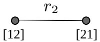
- Γ_n *is connected*
- Γ_n *is* $(n-1)$*–regular*
- Γ_n *is vertex–transitive*
- Γ_n *has a hierarchical structure*
- Γ_n *is hamiltonian*

Star and Pancake graphs: hierarchical structure

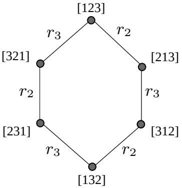
Γ_n consists of n copies $\Gamma_{n-1}(i) = (V^i, E^i)$, $1 \leqslant i \leqslant n$, where the vertex set V^i is presented by permutations with the fixed last element.

P_1

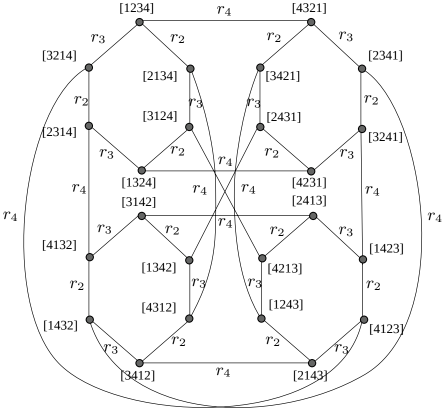
P_2



P_3



P_4



Hamiltonicity

Hamiltonian graph

A graph is hamiltonian if it contains a hamiltonian cycle.

Testing whether a graph is hamiltonian is an NP-complete problem.

Lovász conjecture, 1970

Every connected vertex–transitive graph has a hamiltonian path.

Folk conjecture

Every connected Cayley graph on a finite group has a hamiltonian cycle.

It is true for abelian groups.

Hamiltonicity: Star graph

$Kompel'makher$, $Liskovets$, 1975

The graph $Cay(Sym_n, T)$ is hamiltonian whenever T is a generating set for Sym_n consisting of transpositions.

This result has been generalized as follows.

$Tchuente$, 1982

Let T be a set of transpositions that generate Sym_n. Then there is a hamiltonian path in the graph $Cay(Sym_n, T)$ joining any permutations of opposite parity.

Thus, all transposition Cayley graphs are hamiltonian, hence the Star graph is also hamiltonian.

Hamiltonicity: Pancake graph

Zaks, 1984

The generating algorithm for permutations from which it follows that P_n, $n \geqslant 3$, is hamiltonian, i.e. there is a cycle of length $n!$.

Kanevsky, Feng, 1995

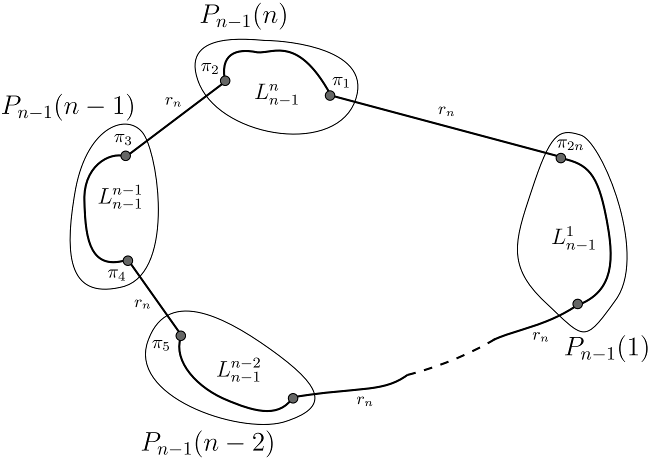
All cycles of length l where $6 \leqslant l \leqslant n! - 2$, or $l = n!$ can embedded in P_n.

Thus, the Pancake graph is also hamiltonian.

Sheu, Tan, Chu, 2006

All cycles of length l where $6 \leqslant l \leqslant n!$ can embedded in P_n.

Hamiltonicity based on the hierarchical structure of the Pancake graph



Automorphism group: Star and Pancake graphs

Feng, 2006

The automorphism group of $Cay(Sym_n, T)$ with a minimal generating set is the semiproduct $R(Sym_n) \bowtie Aut(Sym_n, T)$, where $R(Sym_n)$ is the right regular representation of Sym_n,and
$Aut(Sym_n, T) = \{\alpha \in Aut(Sym_n) | T^\alpha = T\}$.

Feng's result gives the automorphism group for the Star graph.

Deng, _Zhang_, 2012

The automorphism group of the Pancake graph $P_n, n \geqslant 5$, is the left regular representation of the symmetric group Sym_n.

Perfect codes: Star and Pancake graphs

Perfect codes

An independent set D of vertices in a graph Γ is an efficient dominating set (or perfect code) if each vertex not in D is adjacent to exactly one vertex in D.

Dejter, *Serra*, 2002

Existence of efficient dominating sets in Cayley graphs having hierarchical structure (hypercube, Star graph, Pancake graph).

Konstantinova, *Savin*, 2010, 2012

There are n efficient dominating sets in $\Gamma_n \in \{S_n, P_n\}$ given by
$D_k = \{[k\,\pi_2\,\ldots\,\pi_n],\, \pi_j \in \{1,\ldots,n\} \backslash \{k\} : 2 \leqslant j \leqslant n\}, \quad 1 \leqslant k \leqslant n.$

Diameter: Star graph

Akers, Krishnamurthy, 1989

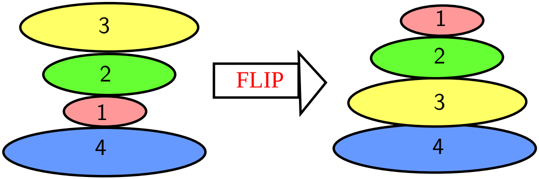
The diameter of the Star graph is $\lfloor \frac{3(n-1)}{2} \rfloor$. *Moreover,*

$$diam(S_n) = \begin{cases} \frac{3(n-1)}{2}, & \text{if } n \text{ odd}, \\ 1 + \frac{3(n-2)}{2}, & \text{if } n > 3 \text{ even}. \end{cases}$$

Remark: The Star graph has a simple cycle structure (only even cycles) which allows to get its diameter.

Diameter: Pancake graph and Pancake problem (*Goodman*, 1975)

"The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several pancakes from the top and flips them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?"



Diameter: Pancake graph and Pancake problem

A stack of n pancakes is represented by a permutation on n elements and the problem is to find the least number of flips (prefix–reversals) needed to transform a permutation into the identity permutation.

This number of flips corresponds to the diameter D of the Pancake graph

The table of diameters for P_n, $4 \leqslant n \leqslant 19$, is presented below:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

Pancake problem: bounds

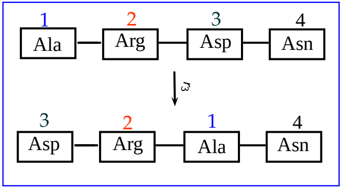
1979, Gates, Papadimitriou: $17n/16 \leqslant D \leqslant (5n+5)/3$

1997, Heydari, Sudborough: $15n/14 \leqslant D$

2007, Sudborough, etc.: $D \leqslant 18n/11$

Applications: molecular biology

Genomes are presented by a permutations:



The evolutionary distance: Palmer, Herbon, 1986

The prefix–reversal distance of two permutations is the least number d of prefix–reversals needed to transform one permutation into another:

$$X\colon (\underline{1,5,2},3,4) \longrightarrow Y\colon (2,5,1,3,4)$$

Sorting permutations by reversal (prefix–reversals): NP–hard

Find, for a given permutation π, a minimal sequence d of reversals (prefix–reversals) that transforms π to the identity permutation I.

Applications: interconnection networks

1986, SIAM International Conference on Parallel Processing: "to use Cayley graphs as a tool to construct vertex–symmetric interconnection networks."

Interconnection networks are modeled by graphs: the vertices correspond to processing elements, memory modules, or just switches; the edges correspond to communication lines.

Advantages in using Cayley graphs as network models:

- *vertex–transitivity (the same routing algorithm is used for each v);*
- *hierarchical structure (allows recursive constructions);*
- *high fault tolerance (the maximum number of vertices that need to be removed and still have the graph remain connected);*
- *small degree and diameter.*

Star graphs $\equiv$ Star networks, Pancake graphs $\equiv$ Pancake networks

Colouring: Cayley graphs

The smallest number of colors needed to color a graph (such that no two adjacent vertices share the same color) is called its chromatic number.

Babai, 1978

Every group has a Cayley graph of chromatic number $\leqslant \omega$; *for solvable groups the minimum chromatic number is* $\leqslant 3$.

Graham, *Grötshel*, *Lovász*(*Eds*.), "Handbook of Combinatorics",1995

Every finite group has a Cayley graph of chromatic number $\leqslant 4$.

Remark: This is a consequence of the fact that every finite simple group is generated by $\leqslant 2$ elements.

Colouring: Star and Pancake graphs

Chromatic number of the Star graph

$\lambda(S_n) = 2$

Chromatic number of the Pancake graph

$\lambda(P_n) = ?$

Pancake graph: known facts

$\lambda(P_2) = \lambda(P_3) = 2, \quad \lambda(P_4) = \lambda(P_5) = 3, \quad 3 \leqslant \lambda(P_6) \leqslant 4.$
$\lambda(P_n) \leqslant n - 2$ *for any* $n > 4$.

Pancake graph: conjecture

$\lambda(P_n) = 3$ *for some* $n > 5$.