Spectral theory of Deza graphs

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The talk is based on joint works with:

Vladislav V. Kabanov, Krasovskii Institute of Mathematics and Mechanics, Yekaterinburg, Russia

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Amir Hossein Ghodrati, Shahid Rajaee Teacher Training University, Tehran, Iran

Willem H. Haemers, Tilburg University, The Netherlands

Saieed Akbari and Mohammad Ali Hosseinzadeh

(February 1, 2020)

Outline of talk

The main goals:

$\diamondsuit$ to show a historical development from strongly regular graphs to (strongly) Deza graphs

$\diamondsuit$ to present a spectral characterization of (strongly) Deza graphs

Content

$\diamond$ Strongly regular graphs: historical background (1963)

$\diamond$ Deza graphs: generalization of strongly regular graphs (1994, 1999)

$\diamond$ Strongly Deza graphs: a new concept (2021)

Strongly regular graphs

Definition

Let $G = (V, E)$ be a regular graph with n vertices and degree k. Then G is a strongly regular graph if:
every two adjacent vertices have λ common neighbours
every two non-adjacent vertices have μ common neighbours.

Notation

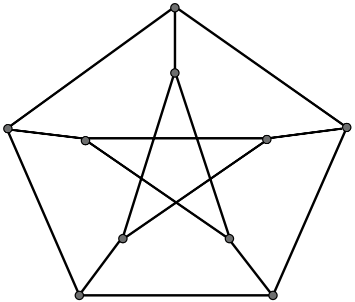
A strongly regular graph is denoted as $srg(n, k, \lambda, \mu)$ (or SRG).

Remark: the original (and more standard) notation is $srg(v, k, \lambda, \mu)$.

Original paper by Raj Chandra Bose, 1963

R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* 13 (1963) 389–419.

Strongly regular graphs: example



Petersen graph is $srg(10, 3, 0, 1)$

Strongly regular graphs: properties

◇ *SRG* is a graph of diameter 2

◇ if $\mu \neq 0$ then the parameters of *SRG* are in the following relationship:

$$n = 1 + k + \frac{k(k-\lambda-1)}{\mu}$$

◇ *SRG* with an adjacency matrix A is represented by matrix equations:

$$AJ = JA = kJ, \qquad (1)$$

where J is the all-ones matrix;

$$A^2 = kI + \lambda A + \mu(J - I - A) \qquad (2)$$

◇ *SRG* has precisely three eigenvalues: k^1, r^f; s^g.

The fact that f, g must be integers is a strong restriction on possible sets.

Strongly regular graphs: spectrum

Theorem (*SRG*-spectral characterization)

Let G be $srg(n, k, \lambda, \mu)$ and the eigenvalues k, r, and s. Then:

(i) *The principal eigenvalue k has the multiplicity 1.*

(ii) *The restricted integer eigenvalues*

$$r, s = \frac{(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

have the multiplicities $f, g = \frac{1}{2}\left(n - 1 \mp \frac{(r + s)(n - 1) + 2k}{r - s}\right)$.

(iii) *If r and s are not integers, then*

$$r, s = \frac{-1 \pm \sqrt{n}}{2}$$

with the same multiplicities.

Strongly regular graphs: spectral properties

SRG-spectral characterization gives us the following fact.

Fact 1

Any *SRG* has precisely three distinct eigenvalues.

Actually, the converse is also true.

Fact 2

Any connected regular graph with only three distinct eigenvalues is *SRG*.

Strongly regular graphs: applications

- ◇ partial geometries (Bose, 1963; van Lint+, 1981; Švob+, 2020)
- ◇ rank 3 permutation groups (Higman, 1964)
- ◇ regular 2-graphs (Higman, Taylor, 1977)
- ◇ local graphs (Hall, 1985)

Main new book

Andries E. Brouwer, Hendrik Van Maldeghem, *Strongly regular graphs*.

`https://homepages.cwi.nl/~aeb/math/srg/rk3/srgw.pdf`

Classical old book

A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989) (Chapter 1).

Deza graphs: generalization of *SRG*

Definition

A Deza graph G with parameters (n, k, b, a) is a k-regular graph of order n for which the number of common neighbours of two vertices takes values b or a, where $b \geqslant a$, whenever G is not the complete or the edgeless graph.

Original paper by Deza & Deza, 1994

A. Deza, M. Deza, The ridge graph of the metric polytope and some relatives. In: *Polytopes: Abstract, Convex and Computational*. NATO ASI Series, Vol. 440 (1994) 359–372, Springer.

Founding Deza graph theory, 1999

M. Erickson, S. Fernando, W.H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, *J. Combinatorial Design*, 7 (1999) 359–405.

Deza graphs: matrix representation

Let G be a graph with n vertices, and M be its adjacency matrix.

Then G is a Deza graph with parameters (n, k, b, a) if and only if

$$M^2 = aA + bB + kI$$

for some symmetric $(0, 1)$-matrices A and B such that

$$A + B + I = J,$$

where J is the all-ones matrix and I is the identity matrix.

Deza graphs: properties

$\diamond$ A Deza graph has diameter at least 2

$\diamond$ Any Deza graph with parameters (n, k, b, a) is a strongly regular graph with parameters (n, k, λ, μ)

Deza graphs: properties

$\diamond$ A Deza graph has diameter at least 2

$\diamond$ Any Deza graph with parameters (n, k, b, a) is a strongly regular graph with parameters (n, k, λ, μ) if and only if

$$M = A, \quad M = B \quad \text{or} \quad b = a.$$

Then we have

$$\{\lambda, \mu\} = \{a, b\}$$

and

$$M^2 = kI + \lambda M + \mu(J - I - M).$$

$\diamond$ A Deza graph of diameter 2 which is not a strongly regular graph is called a *strictly Deza graph*.

http://alg.imm.uran.ru/dezagraphs/dezatab.html

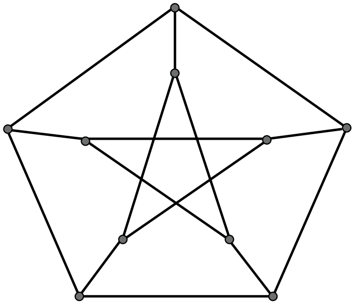
https://arxiv.org/pdf/2102.10624.pdf

Strictly Deza graphs: example

https://conferences.famnit.upr.si/event/13

Strictly Deza graph with parameters $(8, 4, 2, 1)$

Strongly regular graphs - Deza graphs: example



Petersen graph is $srg(10, 3, 0, 1)$ and Deza graph with the same parameters

Deza graphs: children

Definition 1

Let G be a Deza graph with parameters (n, k, b, a) and $b \neq a$. The children G_A and G_B of G are defined as two graphs on the same vertex set $V(G)$ such that for any two distinct vertices $u, v \in V(G)$:

- u, v are adjacent in G_A if and only if the number of their common neighbours is equal to a;
- u, v are adjacent in G_B if and only if the number of their common neighbours is equal to b.

G with parameters $(8, 4, 2, 1) \;\Rightarrow\; G_A \cong C_4 \bigcup C_4$, $G_B \cong K_8 \backslash G_A$

Deza graphs: children

Definition 2

Let G be a Deza graph with M, A and B such that

$$M^2 = aA + bB + kI$$

where

$$A + B + I = J.$$

Then A and B are adjacency matrices of graphs, and the corresponding graphs G_A and G_B are the *children* of G.

Deza graphs and their children: spectrum

Theorem. [Akbari-Ghodrati-Hosseinzadeh-Kabanov-Konstantinova-Shalaginov, Spectra of Deza graphs, *Linear and Multilinear Algebra*, 2020]

Let G be a Deza graph with parameters (n, k, b, a), $b > a$. Let M, A, B be the adjacency matrices of G and its children, respectively. If $\theta_1 = k, \theta_2, \ldots, \theta_n$ are the eigenvalues of M, then

(i) the eigenvalues of A are

$$\alpha = \frac{b(n-1) - k(k-1)}{b-a}, \ \alpha_2 = \frac{k - b - \theta_2^2}{b-a}, \ \ldots, \ \alpha_n = \frac{k - b - \theta_n^2}{b-a}.$$

(ii) the eigenvalues of B are

$$\beta = \frac{a(n-1) - k(k-1)}{a-b}, \ \beta_2 = \frac{k - a - \theta_2^2}{a-b}, \ \ldots, \ \beta_n = \frac{k - a - \theta_n^2}{a-b}.$$

Deza graphs and their children: example

Remark

If $b \neq a$, then children G_A and G_B are regular graphs with degrees $\alpha = \dfrac{b(n-1) - k(k-1)}{b-a}$ and $\beta = \dfrac{a(n-1) - k(k-1)}{a-b}$, respectively.

G with parameters $(8, 4, 2, 1) \;\Rightarrow\; G_A \cong C_4 \bigcup C_4,\ G_B \cong K_8 \backslash G_A$

$$\alpha = \frac{2 \cdot 7 - 4 \cdot 3}{2 - 1} = 2 \text{ and } \beta = \frac{1 \cdot 7 - 4 \cdot 3}{1 - 2} = 5.$$

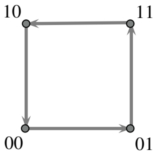
Key open question

The question

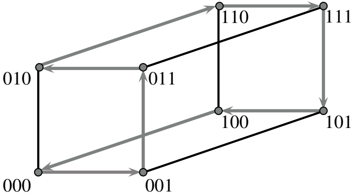
What are multiplicities of Deza graphs (in general)?

Example

The hypercube graph H_n is a Deza graph with parameters $(2^n, n, 2, 0)$. Its spectrum is defined by eigenvalues $n - 2k$ with the multiplicities $\binom{n}{k}$, where $0 \leqslant k \leqslant n$.



H_2



H_3

Deza graphs: generalization of *srg*

From structural point of view:

A Deza graph is a generalization of a strongly regular graph such that: the number of common neighbours of any pair of distinct vertices in a Deza graph does not depend on the adjacency.

From spectral point of view:

- Any strongly regular graph has exactly three distinct eigenvalues.
- A Deza graph can have more than three distinct eigenvalues.

Example

The hypercube graph H_n is a Deza graph with parameters $(2^n, n, 2, 0)$. Its diameter is n. Its spectrum is defined by eigenvalues $n - 2k$ with the multiplicities $\binom{n}{k}$, where $0 \leqslant k \leqslant n$.

Strongly Deza graphs: a new concept

Definition

A *strongly Deza graph* is a Deza graph G with parameters (n, k, b, a) whose children are strongly regular graphs.

First considering, 2021

V. V. Kabanov, E. V. Konstantinova, L. Shalaginov, Generalised dual Seidel switching and Deza graphs with strongly regular children, *Discrete Mathematics*, 344(3) (2021) 112238.
https://doi.org/10.1016/j.disc.2020.112238

Definition and spectral characterization, 2021+

S. Akbari, W. H. Haemers, M. A. Hosseinzadeh, V. V. Kabanov, E. V. Konstantinova, L. Shalaginov, Spectra of strongly Deza graphs, 2021.
https://arxiv.org/abs/2101.06877 (submitted to DM)

Strongly Deza graphs: spectral characterization

By Theorems above, a strongly Deza graph has at most three distinct absolute values of its eigenvalues. But we can be more precise.

Theorem (Spectral characterization-I, AHHKKS-2021+)

Let G be a strongly Deza graph with parameters (n, k, b, a).

(i) *G has at most five distinct eigenvalues.*

(ii) *If G has two distinct eigenvalues, then $a = 0$, $b = k - 1 \geqslant 1$, and G is a disjoint union of cliques of order $k + 1$.*

(iii) *If G has three distinct eigenvalues, then*
- G is a strongly regular graph with parameters (n, k, λ, μ), where $\{\lambda, \mu\} = \{a, b\}$; or
- G is disconnected and each component is a strongly regular graph with parameters (v, k, b, b); or
- each component is a complete bipartite graph $K_{k,k}$ with $k \geqslant 2$.

Strongly Deza graphs: spectral characterization

Theorem (Spectral characterization-II, AHHKKS-2021+)

Let G be a connected Deza graph with parameters (n, k, b, a), $b > a$, and it has at most three distinct absolute values of its eigenvalues.

(i) *If G is a non-bipartite graph, then G is a strongly Deza graph.*

(ii) *If G is a bipartite graph, then either G is a strongly Deza graph or its halved graphs are strongly Deza graphs.*

Definition

If G is a bipartite graph, then the *halved* * graphs of G are two connected components of the graph on the same vertex set, where two vertices are adjacent whenever they are at distance two in G.

* A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989). pp. 25, 438.

Integral strongly Deza graphs

The next theorem gives some conditions on an integral strongly Deza graph with respect to eigenvalues of its children.

Theorem (AHHKKS-2021+)

Let G be a strongly Deza graph with parameters (n, k, b, a). Let its child G_A be a strongly regular graph with parameters $(n, \alpha, \lambda, \mu)$ and eigenvalues α, r, s with multiplicities $1, f, g$. If M is the adjacency matrix of G with spectrum $\{k^1, \theta_2^{m_2}, \theta_3^{m_3}, \theta_4^{m_4}, \theta_5^{m_5}\}$, then one of the statements hold:

(i) $\theta_2^2 = k - b - s(b - a)$ *and* $\theta_3^2 = k - b - r(b - a)$ *are squares; in this case G is an integral graph.*

(ii) $\theta_2^2 = k - b - s(b - a)$ *is not a square; then* $\theta_3^2 = k - b - r(b - a)$ *is a nonzero square and* $m_2 = m_5 = f/2$.

(iii) $\theta_3^2 = k - b - r(b - a)$ *is not a square; then* $\theta_2^2 = k - b - s(b - a)$ *is a nonzero square and* $m_3 = m_4 = g/2$.

This theorem is a generalization of Theorem 2.2 in [HKhM,DDG,2011].

Integral strongly Deza graphs

Integral graph

A graph Γ is *integral* if its spectrum consists entirely of integers.

F. Harary and A. J. Schwenk, Which graphs have integral spectra? *Graphs and Combinatorics* (1974).

The problem of characterizing integral graphs.

Corollary (AHHKKS-2021+)

The children of a strongly Deza graph are integral graphs.

Theorem

Any singular strongly Deza graph is an integral graph with four distinct eigenvalues.*

**singular* graph has zero is its eigenvalue.

Distance-regular (strongly) Deza graphs

A *distance-regular* graph $G = (V, E)$ of order n and diameter d is defined as a connected graph, and there exists numbers $\{a_i, b_i, c_i\}$ such that for every $i \in \{0, \ldots, d\}$ and for every pair of vertices $x \in V$ and $y \in V$ at mutual distance i the following holds:

- the number of vertices adjacent to y at distance $i + 1$ from x equals b_i;
- the number of vertices adjacent to y at distance i from x equals a_i;
- the number of vertices adjacent to y at distance $i - 1$ from x equals c_i.

The numbers a_i, b_i, c_i are called the intersection numbers of G.

k_i is the number of vertices at distance i from a given vertex, $i = 0, \ldots, d$.

E_i is the set of vertex pairs from G at mutual distance i, $i = 0, \ldots, d$.

(V, E_i) is a regular graph of degree k_i, where $E_1 = E$.

Distance-regular (strongly) Deza graphs

Theorem (AHHKKS-2021+)

A distance-regular graph G of diameter $d \geqslant 3$ is a Deza graph if and only if one of the following holds:

(i) $a_1 = 0$*; then* G *has parameters* $(n, k, c_2, 0)$ *and children* (V, E_2) *and its complement.*

(ii) $a_1 = c_2$*; then* G *has parameters* $(n, k, c_2, 0)$ *and children* $(V, E_1 \cup E_2)$ *and its complement.*

$\Longrightarrow$ G is a strongly Deza graph whenever:

1) $a_1 = 0$ and (V, E_2) is strongly regular, or

2) $a_1 = c_2$ and $(V, E_1 \cup E_2)$ is strongly regular.

It is known that the intersection numbers of G determine the eigenvalues of (V, E_1) and $(V, E_1 \cup E_2)$ $\Longrightarrow$ the property 'being a distance-regular strongly Deza graph' is determined by the intersection numbers.

Further research projects

$\diamond$ - characterizing Deza graphs with four and five eigenvalues

$\diamond$ - characterizing isospectral (strongly) Deza graphs

$\diamond$ - characterizing integral Deza graphs

$\diamond$ - studying multiplicities

Further research projects

◇ - characterizing Deza graphs with four and five eigenvalues

◇ - characterizing isospectral (strongly) Deza graphs

◇ - characterizing integral Deza graphs

◇ - studying multiplicities

Thanks for your attention!