Integral graphs: how to get them?

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Outline of the talk

The main goal

To overview known results on integral graphs with emphasizing on ways of constructing integral graphs.

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Content

◇ Historical background

◇ Characterization of:
- cubic integral graphs
- integral trees
- integral Cayley graphs
- quartic integral graphs

◇ Graph operations leading to integral graphs

◇ Open problems

Historical background: 1974

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Remark. We believe our bound is far from being tight and the number of integral graphs is substantially smaller.

Simplest examples

Spectrum of the complete graph K_n

$[(-1)^{n-1}, (n-1)^1]$ for $n \geqslant 2$, and $[0^1]$ for $n = 1$. *Integral for any* $n \geqslant 1$.

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Spectrum of the complete bipartite graph $K_{m,n}$

$[0^{n+m-2}, \pm(\sqrt{nm})^1]$ for $n, m \geqslant 1$. *Integral when* $mn = c^2$.

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Spectrum of n-cycle C_n

The spectrum consists of the numbers $2\cos(\frac{2\pi i}{n})$, $i = 1, \ldots, n$ with multiplicities $2, 1, 1, \ldots, 1, 2$ for n even and $1, 1, \ldots, 1, 2$ for n odd.

There are only three integral cycles:

C_3: $[-1^2, 2]$ $(C_3 \cong K_3)$

C_4: $[-2, 0^2, 2] = [0^2, \pm 2]$ $(C_4 \cong K_{2,2})$

C_6: $[-2, -1^2, 1^2, 2] = [\pm 1^2, \pm 2]$

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Smallest non-integral cycle is C_5: $[2, (\frac{-1+\sqrt{5}}{5})^2, (\frac{-1-\sqrt{5}}{5})^2]$

Classification of integral cubic graphs: 1975-1976

D. Cvetkovič (1975), Cubic integral graphs

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F.C.Bussemaker,D.Cvetkovič (1976);A.J.Schwenk (1978) There are exactly 13 connected, cubic, integral graphs

8 bipartite cubic graphs:

$n = 6 \quad [\pm 3, 0^4]$

$n = 8 \quad [\pm 3, (\pm 1)^3]$

$n = 10 \quad [\pm 3, \pm 2, (\pm 1)^2, 0^2]$

$n = 12 \quad [\pm 3, (\pm 2)^2, \pm 1, 0^4]$ $\quad G_{12} \cong GP(n, 1)$ (Prism graph)

$n = 20 \quad [\pm 3, (\pm 2)^4, (\pm 1)^5]$ $\quad G_9$

$n = 20 \quad [\pm 3, (\pm 2)^4, (\pm 1)^5]$ $\quad G_{10}$

$n = 24 \quad [\pm 3, (\pm 2)^6, (\pm 1)^3, 0^4]$ $\quad G_{13} \cong Cay(\mathrm{Sym}_n, \{(1i)\})$ (Star graph)

$n = 30 \quad [\pm 3, (\pm 2)^9, 0^{10}]$

Classification of integral cubic graphs: isospectral non-isomorphic graphs G_9 and G_{10}

UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 544 — № 576 (1976), 43—48.

552. **THERE ARE EXACTLY 13 CONNECTED, CUBIC, INTEGRAL GRAPHS***

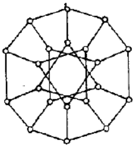
F. C. Bussemaker and *D. M. Cvetković***

1. Results. A graph is called integral if its spectrum consists entirely of integers. Cubic graphs are regular graphs of degree 3.

It was proved in [3] that the set I_r of all connected regular integral graphs of a fixed degree r is finite. At the same time the search for cubic integral graphs was begun. Now we complete this work by the following theorem.

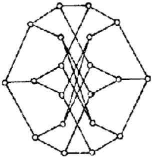
Theorem 1. *There are exactly* 13 *connected, cubic, integral graphs. They are displayed in Fig.* 1 *and in Fig.* 2 *of* [3].

G_9



$3,\ 2^4,\ 1^5,\ -1^5,\ -2^4,\ -3$

G_{10}



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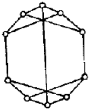
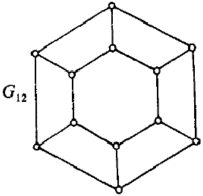
G_{11}



$3,\ 2,\ 1^2,\ 0^2,\ -1^2,\ -2,\ -3$

Classification of integral cubic graphs: Cayley graphs $G_{12} \cong GP(6,1)$ and $G_{13} \cong Cay(\mathrm{Sym}_4, \{(1i)\})$



3, 2^2, 1, 0^4, -1, -2^2, -3

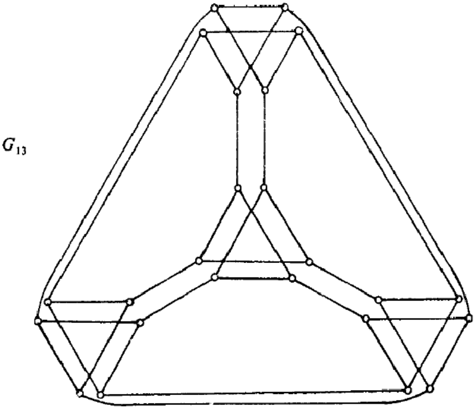


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Fig. 1

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There are exactly 13 connected, cubic, integral graphs

5 non-bipartite cubic graphs:

$n = 4$	$[3, (-1)^3]$	K_4
$n = 6$	$[3, 1, 0^2, (-2)^2]$	
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$n = 10$	$[3, 2, 1^3, (-1)^2, (-2)^3]$	
$n = 12$	$[3, 2^3, 0^2, (-1)^3, (-2)^3]$	$Cay(\mathrm{Alt}_4, \{(123), (312), (12)(34)\})$

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Known facts

Fact 1. If Γ is a bipartite graph, and λ is its eigenvalue with multiplicity $mul(\lambda)$, then $-\lambda$ is also its eigenvalue with the same multiplicity.

Fact 2. If Γ is a r-regular graph, and λ is an eigenvalue of its adjacency matrix, then $|\lambda| \leqslant r$.

Computational results on graphs: 1999-2004

K. Balińska, D. Cvetković, M. Lepović, S. Simić, D. Stevanović, M. Kupczyk, K.T. Zwierzyński, G. Royle

- Brendan McKay's program *GENG* for generating graphs
- Magma
- On-Line Encyclopedia of Integer Sequences, the sequence A064731

http://www.research.att.com/projects/OEIS?Anum=A064731

Connected intergal graphs with $n \leqslant 12$ vertices

n	1	2	3	4	5	6	7	8	9	10	11	12
#	1	1	1	2	3	6	7	22	24	83	236	325

Computational results on trees: 2008

A. Brouwer, Small integral trees

There are only 28 integral trees on at most 50 vertices. There are 105452337029115 09534 nonisomorphic trees on 50 vertices, more than 10^{19}, which shows that integral trees are rare objects.

Connected integral trees with $n \leqslant 50$ vertices and diameter d

n	1	2	5	6	7	10	14	17	19	25	26	31
#	1	1	1	1	1	1	1	3	1	1	3	3
d	0	1	2	3	4	2	3	2,4	4	5	2,3,4	4,6

n	35	37	42	46	49	50
#	1	4	1	1	1	2
d	4	2,4,6	3	4	4	1,4

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- there are integral trees of arbitrarily large even diameter (P. Csikvári, 2010);
- there are infinitely many integral trees of odd diameter (E. Ghorbani, A. Mohammadian, B. Tayfeh-Rezaie, 2011)

Which Cayley graphs are integral?: 2009

Cayley graph

Let G be a group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$. In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the action of the generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$.

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Properties

By the definition,

(i) Γ is undirected with no loops;

(ii) Γ is a connected regular graph of degree $|S|$;

(iii) Γ is a vertex–transitive graph.

Classification of integral cubic Cayley graphs: 2009

A. Abdollahi, E. Vatandoost

There are exactly seven connected cubic integral Cayley graphs. In particular, for a finite group G and a generating set $S, |S| = 3$, the Cayley graph Γ is integral if and only if G is isomorphic to one of the following groups: C_2^2, C_4, C_6, Sym_3, C_2^3, $C_2 \times C_4$, D_8, $C_2 \times C_6$, D_{12}, Alt_4, Sym_4, $D_8 \times C_3$, $D_6 \times C_4$ or $\mathrm{Alt}_4 \times C_2$.

Notation above

C_n is the cyclic group of order n
D_{2n} is the dihedral group of order $2n, n > 2$
Sym_n is the symmetric group of order n
Alt_n is the alternating group of order n

Integral Cayley graphs: 2005-present

Characterization of integral Cayley graphs

- Hamming graphs $H(n,q)$: $\lambda_m = n(q-1) - qm$, where $m = 0, 1, \ldots, n$, with multiplicities $\binom{n}{m}(q-1)^m$
- Cayley graphs over cyclic groups (circulants) (W. So, 2005) (give necessary and sufficient conditions)
- Cayley graphs over abelian groups (W. Klotz, T. Sander, 2010) (determine all abelian Cayley integral groups)
- Cayley graphs over dihedral groups (L. Lu, Q. Huang, X. Huang, 2018) (determine all integral Cayley graphs over D_p for a prime p)

Definition

A group G is a Cayley integral group if for every symmetric subset S of G, $\Gamma = Cay(G,S)$ is an integral graph.

Integral quartic graphs: 1998-present

Known results

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17 quartic bipartite Cayley graphs

n	8	10	12	16	18	24	30	32	36	40	48	72	120
#	1	1	2	1	1	3	1	1	1	1	1	2	1

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users.monash.edu.au/~iwanless/data/graphs/IntegralGraphs

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- 4-regular: the largest eigenvalue is 4 with multiplicity 1;
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- 4-regular bipartite: $n \leqslant 6560$ and $[4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4]$

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- let q, h are the numbers of C_4, C_6; since the sum of the k-th powers of the eigenvalues is just the number of closed walks oh length k, the parameters n, x, y, z, w, q, h satisfy the following Diophantine equations:

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$\frac{1}{2}\sum \lambda_i^0 = 1 + x + y + z + w = n$

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$$\frac{1}{2}\sum \lambda_i^0 = 1 + x + y + z + w = n$$
$$\frac{1}{2}\sum \lambda_i^2 = 16 + 9x + 4y + z = 4n$$

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$$\frac{1}{2}\sum \lambda_i^0 = 1 + x + y + z + w = n$$
$$\frac{1}{2}\sum \lambda_i^2 = 16 + 9x + 4y + z = 4n$$
$$\frac{1}{2}\sum \lambda_i^4 = 256 + 81x + 16y + z = 28n + 4q$$

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- let q, h are the numbers of C_4, C_6; since the sum of the k-th powers of the eigenvalues is just the number of closed walks oh length k, the parameters n, x, y, z, w, q, h satisfy the following Diophantine equations:

$$\frac{1}{2}\sum \lambda_i^0 = 1 + x + y + z + w = n$$
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Resume

As the result, all 32 connected 4-regular integral Cayley graphs are listed.

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Open question

Are there any other graph operations preserving the integrality?