Some problems on Cayley graphs in Computer Science

Elena Konstantinova

Sobolev Institute of Mathematics, Mathematical Center in Akademgorodok, Novosibirsk State University
Three Gorges Mathematical Research Center, China Three Gorges University

e_konsta@math.nsc.ru

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Outline of the talk

Cayley graphs

- definition, properties
- historical background
- classical problems

Cayley graphs as Networks in Computer Science

- Gray codes as checking hamiltonicity
- Structural chracterization of networks
- Integral networks

Cayley graph: definition

Definition

Let G be a group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$. In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the action of the generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$.

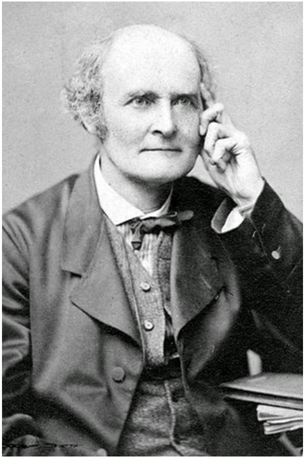
Properties

By the definition, Cayley graph is an ordinary graph: its edges are not oriented and it does not contain loops. Moreover:
(i) Γ is a connected regular graph of degree $|S|$;
(ii) Γ is a vertex–transitive graph.

A graph is vertex-transitive if its automorphism group acts transitively upon its vertices.

Cayley graph: historical background

The definition was introduced by A. Cayley in 1878 to explain the concept of abstract groups which are generated by a set of generators.



http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Cayley.html

Cayley graph: examples

Complete graph

is the Cayley graph for the additive group $\mathbb{Z}_n$ of integers modulo n whose generating set is the set of all non–zero elements of $\mathbb{Z}_n$.

Example

Let $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $S = \{1, 2, 3, 4, 5\}$, then $\Gamma = Cay(G, S) \cong K_6$.

Circulant

is the Cayley graph $Cay(\mathbb{Z}_n, S)$ where $S \subset \mathbb{Z}_n$ is an arbitrary generating set. The most prominent example is the cycle C_n.

Example

Let $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $S = \{1, 5\}$, then $\Gamma = Cay(G, S) \cong C_6$.

Cayley graph: examples

Cayley graph $Cay(\mathbb{Z}_8, \{1, 3, 5, 7\})$

Graphs and Groups, Geometries and GAP (G2G2)

`https://conferences.famnit.upr.si/event/13`

Problems on Cayley graphs

Classical problems

- classification
- enumeration
- structural characterization (chromatic properties, independent sets)
- isomorphism problem (the computational problem of determining whether two finite graphs are isomorphic; it is not known to be solvable in polynomial time nor to be NP-complete)
- diameter problem (computing the diameter of an arbitrary Cayley graph over a set of generators is NP–hard)
- Hamiltonian problem (testing whether a graph is Hamiltonian is an NP-complete problem)
- etc.

All the problems above appear whenever Cayley graphs are considered as models for interconnection networks in computer science.

Applications in Computer science

SIAM International Conference on Parallel Processing, 1986

it was suggested to use Cayley graphs as *a tool to construct vertex–symmetric interconnection networks*

Interconnection networks are modelled by graphs: the vertices correspond to processing elements, memory modules, or just switches; the edges correspond to communication lines.

Advantages in using Cayley graphs as network models

- vertex–transitivity (the same routing algorithm is used for each vertex)
- edge–transitivity (every edge in the graph looks the same)
- hierarchical structure (allows recursive constructions)
- high fault tolerance (the maximum number of vertices that need to be removed and still have the graph remain connected)
- small degree and diameter

Some problems in Computer Science

- Constructing Gray codes as checking hamiltonicity (related to generating combinatorial objects and sorting *smth* by *smth*)
- Structural chracterization of networks (cycles, independent sets, chromatic properties)
- Spectral characterizion of integral networks (as supporting the so-called perfect state transfer)

Gray codes and Hamiltonicity of Cayley graphs

Gray code [F. Gray, 1953, U.S. Patent 2,632,058]

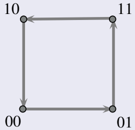
The reflected binary code, also known as Gray code, is a binary numeral system where two successive values differ in only one bit.

Examples

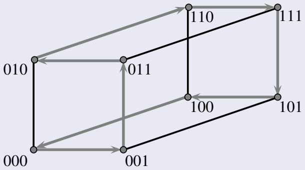
$n = 2$: 00 01 | 11 10

$n = 3$: 000 001 011 010 | 110 111 101 100

RBC is associated with Hamiltonian cycles of Hypercube graphs



H_2



H_3

Gray codes and Hamiltonicity of Cayley graphs

Hypercube graphs are Cayley graphs

$H_n = Cay(\mathbb{Z}_2^n, S)$, where $S = \{(\underbrace{0,\ldots,0}_{i},1,\underbrace{0,\ldots,0}_{n-i-1}), 0 \leqslant i \leqslant n-1\}$.

Folk conjecture,1970

Every Cayley graph on a finite group has a Hamiltonian cycle.

Gray codes: generating combinatorial objects

Now the term Gray code refers to

minimal change order of combinatorial objects.

D.E. Knuth, The Art of Computer Programming, Vol.4, 2010

Gray codes are related to

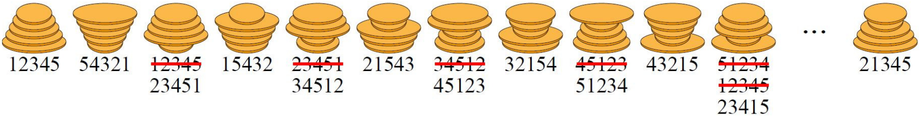
efficient algorithms for exhaustively generating combinatorial objects.

(tuples, permutations, combinations, partitions, trees)

A. Williams, J. Sawada, Greedy pancake flipping, 2013

Take a stack of pancakes, numbered $1, 2, ..., n$ by increasing diameter, and repeat the following:

Flip the maximum number of topmost pancakes that gives a new stack.



Greedy Pancake Gray code over Sym_4:

$$[\overline{1234}],\ [\overline{432}1],\ [\overline{2341}],\ [\overline{143}2],\ [\overline{3412}],\ [\overline{214}3],\ [\overline{4123}],\ [\overline{32}14],$$

$$[\overline{2314}],\ [\overline{413}2],\ [\overline{3142}],\ [\overline{241}3],\ [\overline{1423}],\ [\overline{324}1],\ [\overline{4231}],\ [\overline{13}24],$$

$$[\overline{3124}],\ [\overline{421}3],\ [\overline{1243}],\ [\overline{342}1],\ [\overline{2431}],\ [\overline{134}2],\ [\overline{4312}],\ [\overline{21}34].$$

Background: Pancake problem (*Goodman*, 1975)

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several pancakes from the top and flips them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?

Pancake problem and Pancake graph

Pancake problem

A stack of n pancakes is represented by a permutation on n elements and the problem is to find the least number of flips (prefix–reversals) needed to transform a permutation into the identity permutation. This number of flips corresponds to the diameter D of the Pancake graph.

Pancake graph

P_n is the Cayley graph on the symmetric group $\mathrm{Sym_n}$ with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where r_i is a permutation reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\underline{\pi_1 \ldots \pi_i}\pi_{i+1} \ldots \pi_n]r_i = [\underline{\pi_i \ldots \pi_1}\pi_{i+1} \ldots \pi_n]$.

The table of diameters D for P_n, $4 \leqslant n \leqslant 19$, is presented below:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

Pancake graphs: good properties to be Networks

Known bounds on diameter:

1979, *Gates, Papadimitriou*: $17n/16 \leqslant D \leqslant (5n+5)/3$
1997, *Heydari, Sudborough*: $15n/14 \leqslant D$
2007, *Sudborough, etc.*: $D \leqslant 18n/11$

The Pancake graph is Hamiltonian:

All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.

Cubic Pancake graphs: Pancake Networks

The cubic Pancake graphs are defined as Cayley graphs over the symmetric group Sym_n with generating set of three prefix-reversals.

In [J. Sawada, A. Williams, Successor rules for flipping pancakes and burnt pancakes, *Theoretical Computer Science*, **609** (2016) 60–75] these graphs are called Pancake networks.

Conjectures/open problems on Pancake networks

Sawada-Williams Conjecture 1, 2016

$Cay(\mathrm{Sym_n}, \{r_n, r_{n-1}, r_{n-2}\})$ is hamiltonian.

Computational results confirm this conjecture for small $n = 5, 6, 7, 8$.

Open Problem 1

To characterize all generating sets of cubic Pancake networks and check their hamiltonicity.

Some generating sets of cubic Pancake graphs were found recently in:

Elena V. Konstantinova, Son En Gun, The girths of the cubic Pancake graphs, 2022, https://arxiv.org/abs/2201.05733.

Other useful references:
Konstantinova-Medvedev-2016: Independent even cycles in the Pancake graph and greedy Prefix-reversal Gray codes, Graphs and Combinatorics.

Konstantinova-Medvedev-2014, Small cycles in the Pancake graph, AMC.

Conjectures/open problems on Pancake networks

B_n is a group of signed permutations also known as hyperoctahedral group.

Sawada-Williams Conjecture 2, 2016

The burnt Pancake network $Cay(\mathrm{B_n}, \{r_n^b, r_{n-1}^b, r_{n-2}^b\})$ is hamiltonian.

Nothing known about computational results in this case. In general case, the cyclic structure of the burnt Pancake graphs was studied in:

S. A. Blanco, Ch. Buehrle, A. Patidar, Cycles in the burnt pancake graph, Discrete Applied Mathematics, 271 (2019) 1-14.

Open Problem 2

To characterize all generating sets of cubic burnt Pancake networks and check their hamiltonicity.

Going to the next problems....

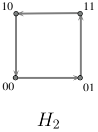
Chromatic properties of Cayley graphs with hierarchical structure

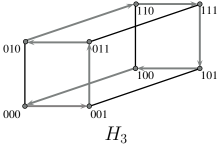
Hierarchical structure

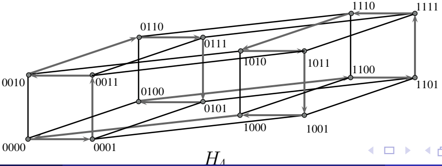
A graph Γ_n has its induced subgraphs as Γ_{n-1} which are vertex-disjoint.

Example

The hypercube graph H_n has two induced subraphs H_{n-1}.







Chromatic number

A mapping $c : V(\Gamma) \to \{1, 2, \ldots, k\}$ is a *proper k–coloring* of a graph $\Gamma = (V, E)$ if $c(u) \neq c(v)$ whenever the vertices u and v are adjacent.

Chromatic number

The chromatic number $\chi(\Gamma)$ of a graph Γ is the least number of colors needed to color vertices of Γ.

A k–coloring is the same as a partition of $V(\Gamma)$ into k independent sets.

Trivial facts

$\chi(Cay(\mathrm{Sym_n}, T)) = 2$ for any $n \geqslant 2$ whenever T is a generating set of transpositions.

Trivial facts

$\chi(H_n) = 2$ for any $n \geqslant 2$.

Other chromatic characteristics of graphs

Chromatic index

The chromatic index $\chi'(\Gamma)$ is the least number of colors needed to color edges of Γ such that no two adjacent edges share the same color.

By Vizing's theorem, the number of colors needed to edge color a simple graph is either its maximum degree Δ (class 1) or $\Delta + 1$ (class 2).

Total chromatic number

The total chromatic number $\chi''(\Gamma)$ of a graph Γ is the least number of colors needed in any total coloring of Γ.

In the *total coloring* no adjacent vertices, edges, and no edge and its endvertices are assigned the same color.

Remark

Edge colorings have applications in scheduling problems and in frequency assignment for fiber optic networks.

Chromatic properties of the Pancake graph [K17]

Total chromatic number

$\chi''(P_n) = n$ for any $n \geqslant 3$.

Total n-coloring is based on efficient dominating sets in the graph.

Chromatic index: class 1

$\chi'(P_n) = n - 1$ for any $n \geqslant 3$.

It is obtained from Vizing's bound $\chi' \geqslant \Delta$ taking into account the edge coloring in which the color $(i-1)$ is assigned to r_i, $2 \leqslant i \leqslant n$.

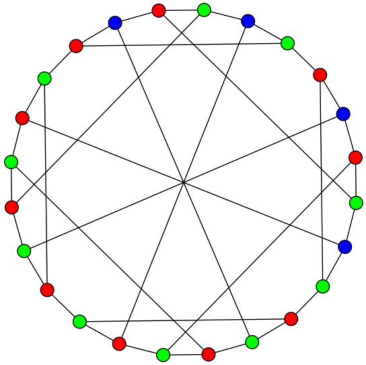
Chromatic number: trivial bounds

$$3 \leqslant \chi(P_n) \leqslant n - 1 \text{ for any } n \geqslant 4.$$

[K17] E. V. Konstantinova, Chromatic properties of the Pancake graphs, Discussiones Mathematicae Graph Theory, **37** (2017) 777–787.

The Pancake graph: chromatic number

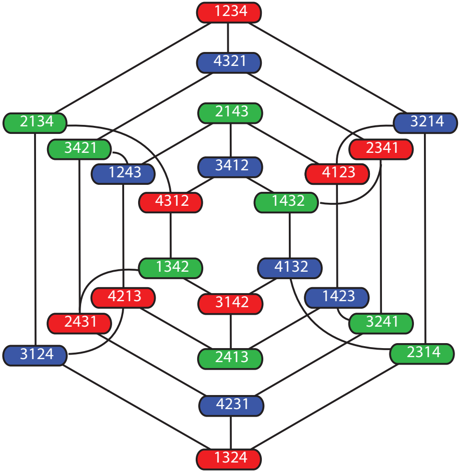
$3 \leqslant \chi(P_n) \leqslant n-1$ for any $n \geqslant 4$.



3-coloring of P_4: hamiltonian drawing

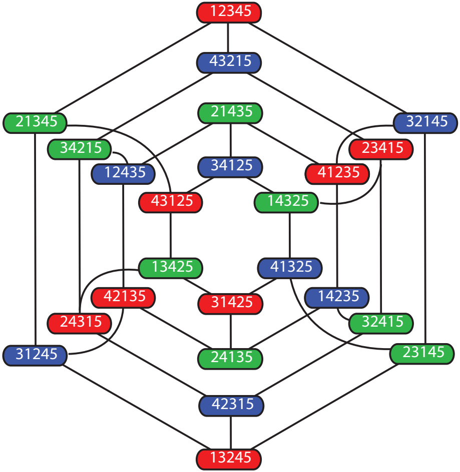
Picture: Tomo Pisanski

3-coloring of P_4: hierarchical drawing



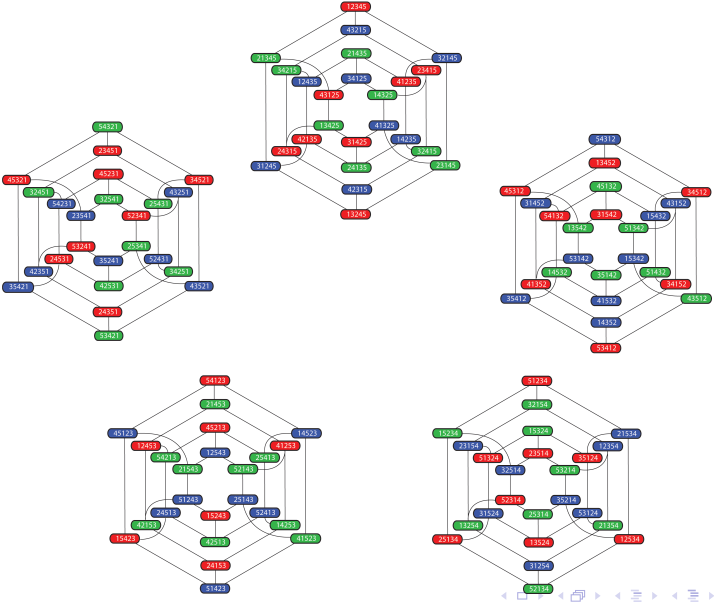
Picture: K. Rogalskaya

3-coloring of one copy of P_5: hierarchical drawing



Picture: K. Rogalskaya

3-coloring P_5: hierarchical drawing



New bound: joint work with Leen Droogendijk

In the table below the known chromatic numbers are presented:

n	3	4	5	6	7	8	9
$\lvert V(P_n)\rvert$	6	24	120	720	5040	40320	362880
$\chi(P_n)$	2	3	3	4	4	4	4

New bound [DK-2021+]

$$\chi(P_n) \leqslant 4\left\lfloor \frac{n}{9} \right\rfloor + \chi\left(P_{n \pmod 9}\right)$$

with $\chi(P_0) = 0$, $\chi(P_1) = 1$, and $\chi(P_2) = 2$. For $n = 3, \ldots, 9$, $\chi(P_n)$ can be taken from the table above. The proof is a consequence of the property.

Subadditive property

$\chi(P_{n+m}) \leqslant \chi(P_n) + \chi(P_m)$ for all positive integers n and m.

With $\chi(P_9) = 4$ this immediately gives the new general upper bound.

Chromatic number: open problems

Open problem 3

It is unknown if $\chi(P_n)$ ever exceeds 4 for $n \geqslant 10$.

If it turns out that $\chi(P_n) \leqslant 4$ for all n, then efforts on finding upper bounds are pointless.

One of the ways to ruling out the possibility $\chi(P_n) \leqslant 4$ for all n would be

Open problem 4

To show that the Pancake graph P_n does not have an independent set of size $\frac{n!}{4}$ if n is large enough.

Equitable chromatic number

A graph is equitably k-colorable if it has a proper k-coloring such that the sizes of any two color classes differ by at most one. The $\chi_=(\Gamma)$ is the smallest integer k such that Γ is equitably k-colorable.

Meyer conjecture, 1973

Every connected graph with maximum degree Δ has an equitable coloring with Δ or fewer colors, with the exceptions of complete graphs, odd cycles.

Meyer conjecture is true for P_n, $n \geqslant 3$.

Droogendijk-Konstantinova conjecture, 2021

For any $n \geqslant 3$, $\chi(P_n) = \chi_=(P_n)$.

DK conjecture confirmed

for any $n = 3, 4, 5, 6, 7$, we have $\chi(P_n) = \chi_=(P_n)$.

On a way to the next problem....

Integral graphs: historical background, 1974

Integral graph

A graph Γ is *integral* if its spectrum consists entirely of integers, where the spectrum of Γ is the spectrum of its adjacency matrix.

F. Harary and A. J. Schwenk, Which graphs have integral spectra? *Graphs and Combinatorics* (1974).

The problem of *characterizing integral graphs*.

O. Ahmadi, N. Alon, I. F. Blake, and I. E. Shparlinski, Graphs with integral spectrum, (2009)

Most graphs have nonintegral eigenvalues, more precisely, it was proved that the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for a sufficiently large n.

Remark. Integral graphs play an important role in quantum networks since a perfect state transfer is supported by such the graphs.

Computational results on graphs: 1999-2004

K. Balińska, D. Cvetković, M. Lepović, S. Simić, D. Stevanović, M. Kupczyk, K.T. Zwierzyński, G. Royle

- Brendan McKay's program *GENG* for generating graphs
- Magma
- On-Line Encyclopedia of Integer Sequences, the sequence A064731

`http://www.research.att.com/projects/OEIS?Anum=A064731`

Connected intergal graphs with $n \leqslant 12$ vertices

n	2	3	4	5	6	7	8	9	10	11	12
total	2	2^3	2^6	2^{10}	2^{15}	2097152	2^{28}	2^{36}	2^{45}	2^{50}	2^{66}
#	1	1	2	3	6	7	22	24	83	236	325

Integral graphs: simplest examples

Spectrum of the complete graph K_n

$[(-1)^{n-1}, (n-1)^1]$ for $n \geqslant 2$, and $[0^1]$ for $n = 1$. *Integral for any* $n \geqslant 1$.

Spectrum of the complete bipartite graph $K_{m,n}$

$[0^{n+m-2}, \pm(\sqrt{nm})^1]$ for $n, m \geqslant 1$. *Integral when* $mn = c^2$.

Spectrum of n-cycle C_n

The spectrum consists of the numbers $2\cos(\frac{2\pi i}{n})$, $i = 1, \ldots, n$ with multiplicities $2, 1, 1, \ldots, 1, 2$ for n even and $1, 1, \ldots, 1, 2$ for n odd.

There are only three integral cycles:

C_3: $[-1^2, 2]$ $\quad (C_3 \cong K_3)$

C_4: $[-2, 0^2, 2] = [0^2, \pm 2]$ $\quad (C_4 \cong K_{2,2})$

C_6: $[-2, -1^2, 1^2, 2] = [\pm 1^2, \pm 2]$

Smallest non-integral cycle is C_5: $[2, (\frac{-1+\sqrt{5}}{5})^2, (\frac{-1-\sqrt{5}}{5})^2]$

Integral Cayley graphs: 2005-present

Characterization of integral Cayley graphs

- Hamming graphs $H(n, q)$: $\lambda_m = n(q-1) - qm$, where $m = 0, 1, ..., n$, with multiplicities $\binom{n}{m}(q-1)^m$
- Cayley graphs over cyclic groups (circulants) (W. So, 2005)
- 3-regular Cayley graphs (A. Abdollahi, E. Vatandoost, 2009)
- Cayley graphs over abelian groups (W. Klotz, T. Sander, 2010)
- spectrum of the Star graph S_n (G. Chapuy, V. Feray, 2012)
- 4-regular Cayley graphs (M. Minchenko, I. M. Wanless, 2015)
- Cayley graphs over dihedral groups (L. Lu, Q. and X. Huang, 2018)

The Star graph $S_n = Cay(\mathrm{Sym}_n, S)$, $n \geqslant 2$

is the Cayley graph over the symmetric group Sym_n with the generating set $S = \{(1\ i),\ 2 \leqslant i \leqslant n\}$.

Integrality of the Star graphs S_n

Conjecture (A. Abdollahi and E. Vatandoost, 2009)

The spectrum of S_n is integral, and contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an eigenvalue of S_n).

For $n \leqslant 6$, the conjecture was verified by GAP.

Theorem *(G. Chapuy and V. Feray, 2012)*

The spectrum of S_n contains only integers. The multiplicity $mul(n-k)$, where $1 \leqslant k \leqslant n-1$, of an integer $(n-k) \in \mathbb{Z}$ is given by:

$$mul(n-k) = \sum_{\lambda \vdash n} dim(V_\lambda) I_\lambda(n-k),$$

where $dim(V_\lambda)$ is the dimension of an irreducible module, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ, satisfying $c(n) = n-k$.

Multiplicities of eigenvalues of the Star graphs

Corollary *(G. Chapuy and V. Feray, 2012)*

Let $n \geqslant 2$, then for each integer $1 \leqslant k \leqslant n-1$ the values $\pm(n-k)$ are eigenvalues of S_n with multiplicity at least $\binom{n-2}{k-1}\binom{n-1}{k}$. The bound is achieved for $k = 2$.

Theorem *(S. Avgustinovich, E. Khomyakova, E. K., 2016)*

The multiplicities $\mathrm{mul}(n-k)$, where $k = 2, 3, 4, 5$ and $n \geqslant 2k-1$, of the eigenvalues $(n-k)$ of the Star graph S_n are given by the following formulas:

$$\mathrm{mul}(n-2) = (n-1)(n-2)$$
$$\mathrm{mul}(n-3) = \frac{(n-3)(n-1)}{2}(n^2-4n+2)$$
$$\mathrm{mul}(n-4) = \frac{(n-2)(n-1)}{6}(n^4-12n^3+47n^2-62n+12)$$
$$\mathrm{mul}(n-5) = \frac{(n-2)(n-1)}{24}(n^6-21n^5+169n^4-647n^3+1174n^2-820n+60)$$

Multiplicities of eigenvalues of the Star graphs

Theorem *(E. Khomyakova, 2018)*

Let $n, t \in \mathbb{Z}$, $n \geqslant 2$ and $1 \leqslant t \leqslant \frac{n+1}{2}$, then the multiplicity $mul(n-t)$ of the eigenvalue $(n-t)$ of the Star graph S_n is given by the following formula:

$$mul(n-t) = \frac{n^{2(t-1)}}{(t-1)!} + P(n),$$

where $P(n)$ is a polynomial of degree $2t-3$.

Catalogue of the Star graph eigenvalue multiplicities (E. Khomyakova, E. Konstantinova, 2019)

Multiplicities $\mathrm{mul}(n-k)$ of eigenvalues $(n-k)$ of the Star graphs S_n for $n \leqslant 50$ and $1 \leqslant k \leqslant n$ are presented in the catalogue. Negative eigenvalues $-(n-k)$ have the same multiplicities as the corresponding positive ones.

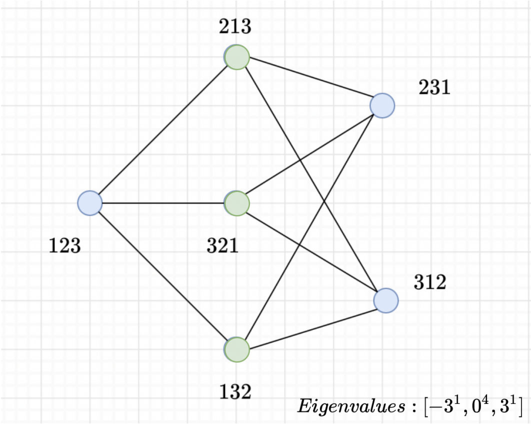
Transposition graph and its properties

Definition

The *Transposition graph* T_n is defined as a Cayley graph over the symmetric group Sym_n generated by all transpositions $T = \{(i\,j) \in Sym_n, 1 \leqslant i < j \leqslant n\}$.

Properties:

- Connected
- Bipartite
- $\binom{n}{2}$-regular
- Order is $n!$
- Integral (Lytkina-K, *Algebra Colloquium* (2020))



K. Kalpakis, Y. Yesha, On the bisection Width of the Transposition network, *Networks*, **29** (1997) 69–76.

Result 1

The Transposition graph $T_n, n \geqslant 2$, is an integral graph such that its largest eigenvalue is $\frac{n(n-1)}{2}$ with multiplicity 1; its second largest eigenvalue is $\frac{n(n-3)}{2}$ with multiplicity $(n-1)^2$; and for each $k, 1 \leqslant k \leqslant n$, the value $\frac{n(n-2k+1)}{2}$ is an eigenvalue of T_n with multiplicity at least $\frac{n!}{n(n-k)!(k-i)!}$.

Remark. Result 1 does not give a complete description of the T_n spectrum.

Result 2

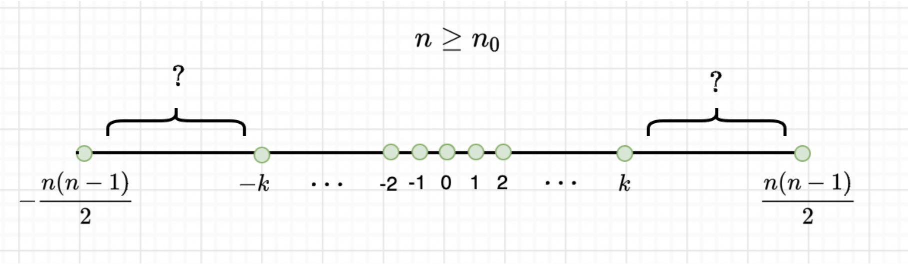
Let $\mathbf{i} = (n_1, \ldots, n_k) \vdash n$ is a partition of n. Then

$$\lambda_{\mathbf{i}} = \sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2} \in Spec(T_n).$$

E. Konstantinova, A. Kravchuk, Spectrum of the Transposition graph, *Linear Algebra and its Applications*, **654** (2022) 379-389.

First eigenvalues of T_n

For any integer $k \geqslant 0$, there exists n_0 such that for any $n \geqslant n_0$ and any $m \in \{0, \ldots, k\}$, $m \in Spec(T_n)$.



The Transposition graph: Eigenvalues around zero

The eigenvalue zero and one [KK-2022]

In the spectrum of T_n there is the eigenvalue zero for any $n \neq 2$ and the eigenvalue one for any odd $n \geqslant 7$ and any even $n \geqslant 14$.

Open questions

What are the multiplicities of the eigenvalues zero and one?
What one can say about their asymptotic behavior?

Computational results for the eigenvalue zero:

n	1	3	4	5	6	7	8	9	10	11
$\mathrm{mul}(0)$	1	4	4	36	256	400	9864	6664	790528	1474848

Further studying: Zero eigenvalues of T_n

Joint project with Saúl A. Blanco, IU, and Charles Buehrle, NDMU.

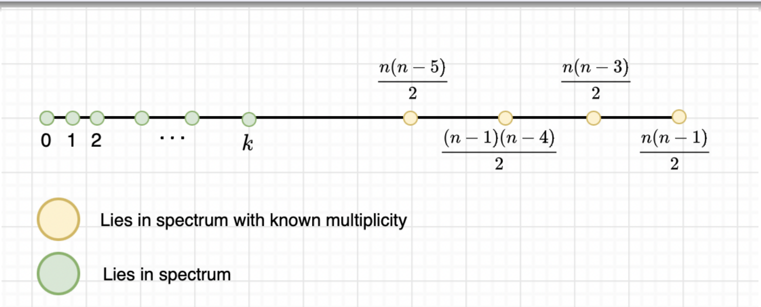
The third and the fourth largest eigenvalues of T_n

Theorem [KK-2022]

The third largest eigenvalue of $T_n, n \geqslant 4$, is $\frac{(n-1)(n-4)}{2}$, and the fourth largest eigenvalue of $T_n, n > 6$, is $\frac{n(n-5)}{2}$.

Open questions

What are multiplicities of small largest eigenvalues? What are expressions for eigenvalues lying between the smallest and largest eigenvalues.



The Transposition graph: spectrum in general

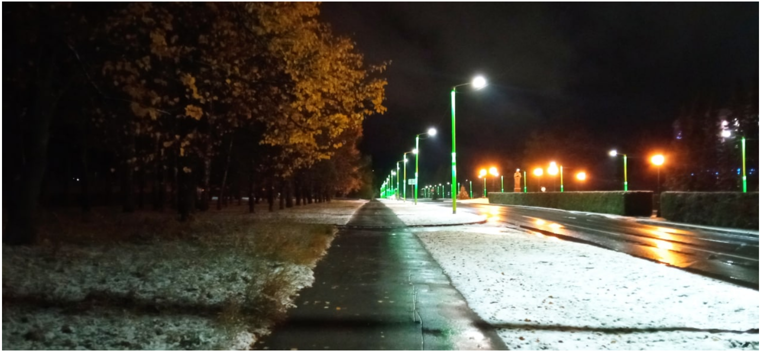
Let $\mathrm{Spec}(\Gamma) = [\lambda_1^{mul(\lambda_1)}, \ldots, \lambda_k^{mul(\lambda_k)}]$ be a spectrum of a graph Γ with non-negative eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ and their multiplicities.

Open problem 5

To describe explicitly the spectrum of T_n.

Some known spectra

$\mathrm{Spec}(T_3) = [0^4, 3^1]$
$\mathrm{Spec}(T_4) = [0^4, 2^9, 6^1]$
$\mathrm{Spec}(T_5) = [0^{36}, 2^{25}, 5^{16}, 10^1]$
$\mathrm{Spec}(T_6) = [0^{256}, 3^{125}, 5^{81}, 9^{25}, 15^1]$
$\mathrm{Spec}(T_7) = [0^{400}, 1^{441}, 3^{1225}, 6^{196}, 7^{225}, 9^{196}, 14^{36}, 21^1]$
$\mathrm{Spec}(T_8) = [0^{9864}, 2^{3136}, 4^{6125}, 7^{4096}, 8^{196}, 10^{784}, 12^{441}, 14^{400}, 20^{49}, 28^1]$



Thanks for your attention!