Small cycles in the Pancake graph

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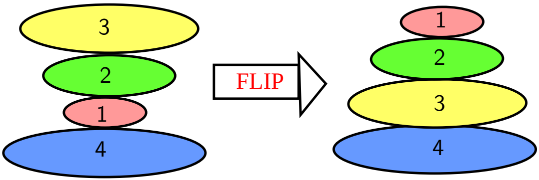
(joint work with A. Medvedev, Novosibirsk State University)

Pancake problem: 1975, *American Mathematical Monthly* by *Jacob E. Goodman* (under the name "Harried Waiter")

"The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several pancakes from the top and flips them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?"



The Pancake problem and the Pancake graph

A stack of n pancakes is represented by a permutation on n elements and the problem is to find the least number of flips (prefix–reversals) needed to transform a permutation into the identity permutation.

Thr Pancake graph: definition

The Pancake graph

$$P_n = (Sym_n, PR)$$

is a Cayley graph on the symmetric group Sym_n with generating set

$$PR = \{r_i \in Sym_n, 1 \leqslant i < n\}, \ |PR| = (n-1),$$

where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e.,

$$[\pi_1, \ldots, \pi_i, \pi_{i+1}, \ldots, \pi_n] r_i = [\pi_i, \ldots, \pi_1, \pi_{i+1}, \ldots, \pi_n].$$

The Pancake problem: still it is open!

This number of flips corresponds to the diameter D of the Pancake graph.

Exact values of D are known for $n \leqslant 19$

4	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
4	*5*	*7*	*8*	*9*	*10*	*11*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*22*

Asai S., Shinano Y., Kaneko K., LNCS 4128 (2006) 1114–1124.
J. Cibulka, Theoretical Computer Science 412 (2011) 822–834.

Pancake problem: bounds

1979, Gates, Papadimitriou: $17n/16 \leqslant D \leqslant (5n+5)/3$
1997, Heydari, Sudborough: $15n/14 \leqslant D$
2007, Sudborough, etc.: $D \leqslant 18n/11$

The main properties of the Pancake graph

Properties

- *a connected $(n-1)$–regular graph of order $n!$;*
- *a vertex–transitive;*
- *a hamiltonian (1984, Zaks);*
- *almost pancyclic graph;*

Hierarchical structure

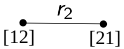
P_n *is constructed from n copies of* $P^i_{n-1} = (V^i, E^i)$, $1 \leqslant i \leqslant n$, *s.t.:*

- $V^i = \{[\pi_1 \ \ldots \ \pi_{n-1} \ i]\}$, $|V^i| = (n-1)!$, $|V(P_n)| = (n-1)!n$;
- $E^i = \{\{[\pi_1 \ \ldots \ \pi_{n-1} \ i], [\pi_1 \ \ldots \ \pi_{n-1} \ i] \, r_j\}, 2 \leqslant j \leqslant n-1\}$, $|E^i| = \frac{(n-1)!(n-2)}{2}$;
- $|E(P_n)| = |E_{in}| \bigcup |E_{ex}|$, $E_{in} = \bigcup E^i$ *и* $|E_{ex}| = \frac{n!}{2}$.

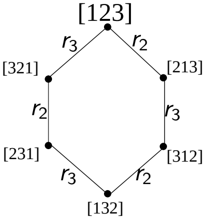
Example: P_1, P_2, P_3, P_4

P_1

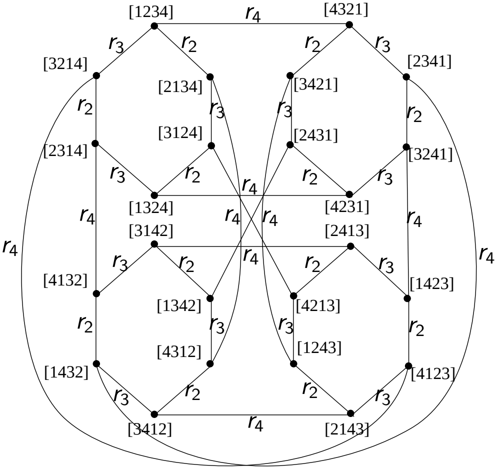
P_2



P_3



P_4



Cycle structure of the Pancake graph

S. Zaks, 1984

P_n, $n \geqslant 3$, is hamiltonian, i.e. there is a cycle of length $n!$

A. Kanevsky, C. Feng, 1995

All cycles of length l where $6 \leqslant l \leqslant n! - 2$, or $l = n!$ can embedded in P_n

J. J. Sheu, J. J. M. Tan, K. T. Chu, 2006

All cycles of length l where $6 \leqslant l \leqslant n!$ can embedded in P_n

However: Explicit description of cycles was not given!

Explicit representation of 6- and 7–cycles in P_n

E. K., A. Medvedev, 2010

1. P_n*,* $n \geqslant 3$*, has* $\frac{n!}{6}$ *independent cycles of length* 6 *described by*

$$C_6 = r_3\, r_2\, r_3\, r_2\, r_3\, r_2,$$

i.e., each of vertices of P_n*,* $n \geqslant 4$*, belongs to the only cycles of length* 6*.*

2. P_n*,* $n \geqslant 4$*, has* $n!\,(n-3)$ *different cycles of length* 7 *described by*

$$C_7 = r_k\, r_{k-1}\, r_k\, r_{k-1}\, r_{k-2}\, r_k\, r_2, \quad 4 \leqslant k \leqslant n,$$

each of vertices of P_n*,* $n \geqslant 4$*, belongs to* $7\,(n-3)$ *cycles of length* 7*.*

3. P_n*,* $n \geqslant 4$*, has* N_7 *independent cycles of length* 7 *where*

$$\frac{n!}{8} \leqslant N_7 \leqslant \frac{n!}{7}.$$

Explicit representation of 8-cycles in P_n

E. K., A. Medvedev, 2011

Each of vertices of P_n, $n \geqslant 4$, belongs to N different 8–cycles of the following 8 canonical forms:

$r_k\, r_j\, r_i\, r_j\, r_k\, r_{k-j+i}\, r_i\, r_{k-j+i}$, $\quad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n;$

$r_k\, r_{k-1}\, r_2\, r_{k-1}\, r_k\, r_2\, r_3\, r_2$, $\quad 4 \leqslant k \leqslant n;$

$r_k\, r_{k-i}\, r_{k-1}\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_i$, $\quad 2 \leqslant i \leqslant k-2,\ 4 \leqslant k \leqslant n;$

$r_k\, r_{k-i+1}\, r_k\, r_i\, r_k\, r_{k-i}\, r_{k-1}\, r_{i-1}$, $\quad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n;$

$r_k\, r_{k-1}\, r_{i-1}\, r_k\, r_{k-i+1}\, r_{k-i}\, r_k\, r_i$, $\quad 3 \leqslant i \leqslant k-2,\ 5 \leqslant k \leqslant n;$

$r_k\, r_{k-1}\, r_k\, r_{k-i}\, r_{k-i-1}\, r_k\, r_i\, r_{i+1}$, $\quad 2 \leqslant i \leqslant k-3,\ 5 \leqslant k \leqslant n;$

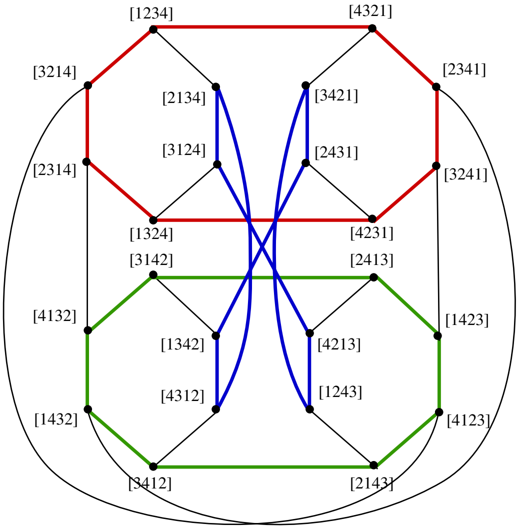
$r_k\, r_{k-j+1}\, r_k\, r_i\, r_k\, r_{k-j+1}\, r_k\, r_i$, $\quad 2 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n;$

$r_4\, r_3\, r_4\, r_3\, r_4\, r_3\, r_4\, r_3$,

$$\textit{where} \quad N = \frac{n^3 + 12n^2 - 103n + 176}{2}.$$

The total number of 8–cycles in the Pancake graph is $\frac{n!\, N}{8}$.

Cycle structure: packing by independent 8–cycles



Cycle structure: packing by independent cycles

Known facts

The Pancake graph P_n, $n \geqslant 3$, has:

- $\frac{n!}{6}$ *independent 6–cycles;*
- $\frac{n!}{8}$ *independent 8–cycles;*
- $\frac{n!}{8} \leqslant N_7 \leqslant \frac{n!}{7}$ *independent 7–cycles.*

Open question

Are there $\frac{n!}{7}$ *independent 7–cycles in* P_n, $n \geqslant 7$*?*

Open question

Are there $\frac{n!}{l}$ *independent l–cycles in* P_n *for some* $n \geqslant 3$*?*

Algebraic representation of cycles in the Pancake graph

The total number of 9–cycles in the Pancake graph is $O(n^3 n!)$

Open question

What is the way to describe all l–cycles in the Pancake graph?

Known facts

Each of vertices of P_n, $n \geqslant 4$, belongs to the following odd cycles

$$C^1_{2s+3} = (r_k\, r_{k-1})^s\, r_{k-s}\, r_k\, r_s, \qquad 2 \leq s \leq k-2,\ 4 \leq k \leq n;$$
$$C^2_{2s+5} = r_k\, (r_{k-1}\, r_{k-2})^s\, r_k\, r_{s+1}\, r_k\, r_{k-s}, \qquad 1 \leq s \leq k-2,\ 4 \leq k \leq n,$$

where $(r_{i_1} \ldots r_{i_j})^s$ corresponds to $s \geq 1$ sequences of $r_{i_1} \ldots r_{i_j}$, $j \geq 2$.