Prefix-reversal Gray codes

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Binary reflected Gray code (BRGC)

Gray code [F. Gray, 1953, U.S. Patent 2,632,058]

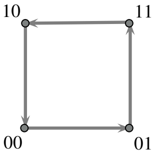
The reflected binary code, also known as Gray code, is a binary numeral system where two successive values differ in only one bit.

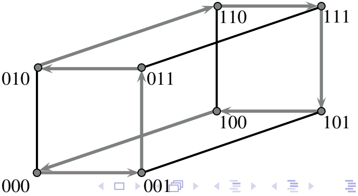
Example

$n = 2$: 00 01 | 11 10

$n = 3$: 000 001 011 010 | 110 111 101 100

BRGC is related to Hamiltonian cycles of hypercube graphs





Gray codes: generating combinatorial objects

Gray codes

Now the term Gray code refers to
minimal change order of combinatorial objects.

[D.E. Knuth, The Art of Computer Programming, Vol.4 (2010)]

Knuth recently surveyed combinatorial generation:

Gray codes are related to
efficient algorithms for exhaustively generating combinatorial objects.

(tuples, permutations, combinations, partitions, trees)

[P. Eades, B. McKay, An algorithm of generating subsets of fixed size with a strong minimal change property (1984)]

They followed to Gray's approach to order
the k–combinations of an n element set.

Gray codes: generating permutations

[V.L. Kompel'makher, V.A. Liskovets, Successive generation of permutations by means of a transposition basis (1975)]

Q: Is it possible to arrange permutations of a given length so that each permutation is obtained from the previous one by a transposition?

A: YES

[S. Zaks, A new algorithm for generation of permutations (1984)]

In Zaks' algorithm each successive permutation is generated by reversing a suffix of the preceding permutation.

Start with $I_n = [12 \dots n]$ *and in each step reverse a certain suffix. Let*

ζ_n *is the sequence of sizes of these suffixes defined by recursively as follows:*

$$\zeta_2 = 2$$
$$\zeta_n = (\zeta_{n-1}\, n)^{n-1}\, \zeta_{n-1},\ n > 2,$$

where a sequence is written as a concatenation of its elements.

Zaks' algorithm: examples

If $n = 2$ then $\zeta_2 = 2$ and we have:

$$[\underline{12}] \quad [21]$$

If $n = 3$ then $\zeta_3 = 23232$ and we have:

$$[1\underline{23}] \quad [2\underline{31}] \quad [3\underline{12}]$$

$$[\underline{132}] \quad [\underline{213}] \quad [321]$$

If $n = 4$ then $\zeta_4 = 23232423232423232423232$ and we have:

$$[12\underline{34}] \quad [23\underline{41}] \quad [34\underline{12}] \quad [41\underline{23}]$$

$$[1\underline{243}] \quad [2\underline{314}] \quad [3\underline{421}] \quad [4\underline{132}]$$

$$[13\underline{42}] \quad [24\underline{13}] \quad [31\underline{24}] \quad [42\underline{31}]$$

$$[1\underline{324}] \quad [2\underline{431}] \quad [3\underline{142}] \quad [4\underline{213}]$$

$$[14\underline{23}] \quad [21\underline{34}] \quad [32\underline{41}] \quad [43\underline{12}]$$

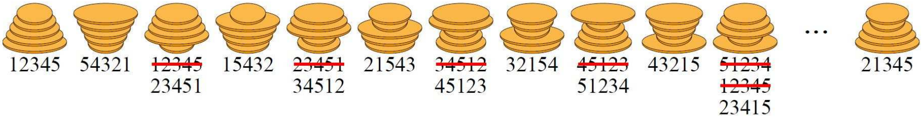
$$[\underline{1432}] \quad [\underline{2143}] \quad [\underline{3214}] \quad [4321]$$

Greedy Pancake Gray codes: generating permutations

[A. Williams, J. Sawada, Greedy pancake flipping (2013)]

Take a stack of pancakes, numbered $1, 2, \ldots, n$ *by increasing diameter, and repeat the following:*

Flip the maximum number of topmost pancakes that gives a new stack.



$[\overline{1234}]$ $[\overline{432}1]$ $[\overline{2341}]$ $[\overline{143}2]$ $[\overline{3412}]$ $[\overline{214}3]$ $[\overline{4123}]$ $[\overline{32}14]$

$[\overline{2314}]$ $[\overline{413}2]$ $[\overline{3142}]$ $[\overline{241}3]$ $[\overline{1423}]$ $[\overline{324}1]$ $[\overline{4231}]$ $[\overline{13}24]$

$[\overline{3124}]$ $[\overline{421}3]$ $[\overline{1243}]$ $[\overline{342}1]$ $[\overline{2431}]$ $[\overline{134}2]$ $[\overline{4312}]$ $[\overline{21}34]$

Prefix–reversal Gray codes: generating permutations

Each 'flip' is formally known as prefix–reversal.

The Pancake graph P_n

is the Cayley graph on the symmetric group Sym_n with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\pi_1 \dots \pi_i \pi_{i+1} \dots \pi_n] r_i = [\pi_i \dots \pi_1 \pi_{i+1} \dots \pi_n]$.

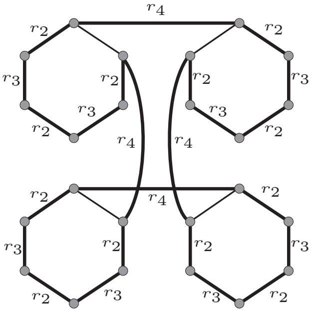
Williams' prefix–reversal Gray code: $(r_n \, r_{n-1})^n$

Flip the maximum number of topmost pancakes that gives a new stack.

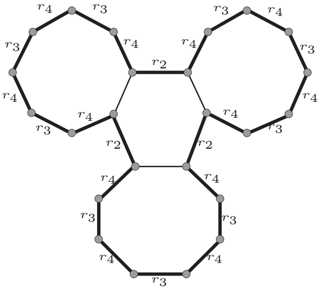
Zaks' prefix–reversal Gray code: $(r_3 \, r_2)^3$

Flip the minimum number of topmost pancakes that gives a new stack.

Two scenarios of generating permutations: Zaks | Williams



(a) Zaks' code in P_4



(b) Williams' code in P_4

Resume: both approaches are based on independent cycles in P_n

Pancake graph: cycle structure

[A. Kanevsky, C. Feng, On the embedding of cycles in pancake graphs (1995)]

All cycles of length ℓ, where $6 \leqslant \ell \leqslant n! - 2$, or $\ell = n!$, can embedded in P_n.

[J.J. Sheu, J.J.M. Tan, K.T. Chu, Cycle embedding in pancake interconnection networks (2006)]

All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can embedded in P_n.

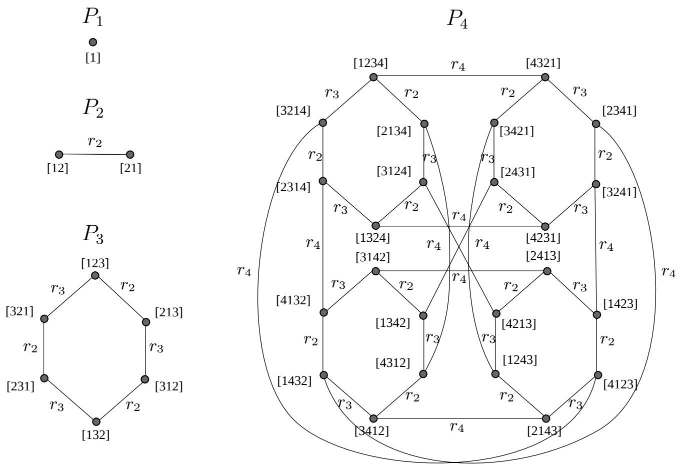
Cycles in P_n

All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.

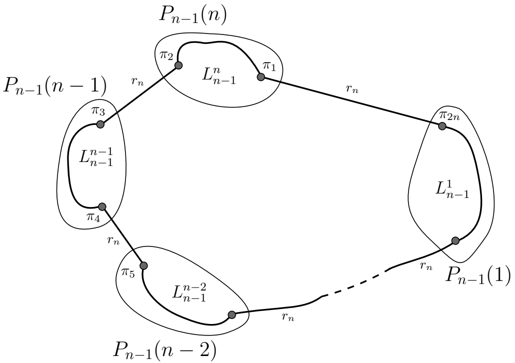
Proofs are based on the hierarchical structure of P_n.

Pancake graphs: hierarchical structure

P_n consists of n copies of $P_{n-1}(i) = (V^i, E^i)$, $1 \leqslant i \leqslant n$, where the vertex set V^i is presented by permutations with the fixed last element.



Hamiltonicity due to the hierarchical structure of $P_n \Leftrightarrow$ Prefix–reversal Gray codes (PRGC) by Zaks and Williams



Proposition 1.

If there is a Gray code in P_{n-1} then

there is a Gray code in P_n given by the same algorithm.

Small independent even cycles and PRGC

Proposition 2.

The Pancake graph $P_n, n \geqslant 3$, contains the maximal set of $\frac{n!}{\ell}$ independent ℓ–cycles of the canonical form

$$C_\ell = (r_k r_{k-1})^k,$$

where $\ell = 2\,k$, for any $3 \leqslant k \leqslant n$.

Williams' prefix–reversal Gray code: $(r_n\, r_{n-1})^n$

This code is based on the maximal set of independent 2n–cycles.

Zaks' prefix–reversal Gray code: $(r_3\, r_2)^3$

This code is based on the maximal set of independent 6–cycles.

Independent cycles in P_n

Theorem 1.

The Pancake graph $P_n, n \geqslant 4$, contains the maximal set of $\frac{n!}{\ell}$ independent ℓ–cycles of the canonical form

$$C_\ell = (r_n\, r_m)^k, \tag{1}$$

where $\ell = 2\,k$, $2 \leqslant m \leqslant n-1$ and

$$k = \begin{cases} O(1) & \text{if } \ m \leqslant \lfloor \frac{n}{2} \rfloor; \\ O(n) & \text{if } \ m > \lfloor \frac{n}{2} \rfloor \ \text{ and } \ n \equiv 0 \pmod{n-m}; \\ O(n^2) & \text{else.} \end{cases} \tag{2}$$

Corollary

The cycles presented in Theorem 1 have no chords.

Hamilton cycles based on small independent even cycles

Hamilton cycle $\Rightarrow$ *PRGC*

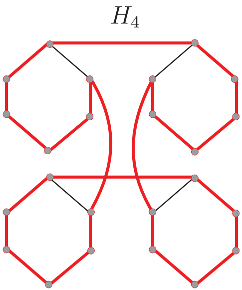
Definition

The Hamilton cycle H_n based on independent ℓ–cycles is called a Hamilton cycle in P_n, consisting of paths of lengths $l = \ell - 1$ of independent cycles, connected together with external to these cycles edges.

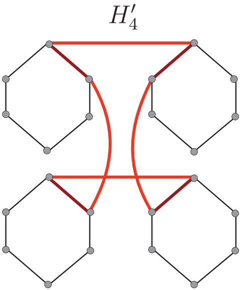
Hamilton cycles based on small independent even cycles

Definition

The complementary cycle H'_n to the Hamilton cycle H_n based on independent cycles is defined on unused edges of H_n and the same external edges.



(c) Hamilton cycle H_4 in P_4



(d) Complement cycle H'_4 to H_4 in P_4

Hamilton cycles based on small independent even cycles

Theorem 2.

There are no other Hamilton cycles in P_n, $n \geqslant 5$, based on independent cycles from Theorem 1 when $k = O(1)$ and $k = O(n)$, except from Zaks and Williams constructions.

Proof is based on examining the complementary cycles' structures.

Hamilton cycles based on independent $\frac{n!}{2}$–cycles

Theorem 3.

There are no Hamilton cycles in P_n, $n \geqslant 4$, based on independent $\frac{n!}{2}$–cycles but there are Hamilton paths based on the following two independent cycles:

$$C_n^1 = ((C_{n-1}^1 / r_{n-1}) r_n)^n,$$
$$C_n^2 = ((C_{n-1}^2 / r_{n-1}) r_n)^n,$$

where $C_4^1 = (r_3\, r_2\, r_4\, r_2\, r_3\, r_4)^2$ *and* $C_4^2 = (r_2\, r_3\, r_4\, r_3\, r_2\, r_4)^2$.

Proof is based on the hierarchical structure of P_n and
on the nonexistence 4–cycles in P_n.

Thanks for attention!